

We have already run into series developments of functions several times: the exponential, sine, cosine functions were expanded into power series; Taylor's theorem provides a way to develop series expansions for suitable functions; the exponential of a matrix gives us the only sure way to "solve" a system of constant coefficient linear equations. We shall see in this chapter that a general technique for solving a differential equation involves approximation of the solution by series expansions.

We shall begin by formulating the definition of convergence of a series of continuous functions and verifying the general criteria guaranteeing convergence. One of the most important of series expansions is that of power series. We shall say that a function is **analytic** if it can be locally developed into a power series. We shall finally verify the fundamental theorem of algebra and complete the discussion of constant coefficient equations. We have delayed this until now because the kind of analytic techniques involved in the fundamental theorem are those which are most appropriately developed for the class of analytic functions. Further techniques for operating with power series will be explored, as well as the question of estimation of the error in replacing the power series by a partial sum.

## 5.1 Convergence

**Definition 1.** Let  $\{f_k\}$  be a sequence of continuous functions defined on a subset  $X$  of  $R^n$ . The **series** formed of the  $\{f_k\}$  is the sequence of partial sums  $\{\sum_{k=1}^n f_k\}$ . We say that the **series converges** if the sequence of these partial sum converges (in the sense of Definition 19 of Chapter 2), and denote the limit by  $\sum_{k=1}^{\infty} f_k$ .

Precisely then,  $f = \sum_{k=1}^{\infty} f_k$  if, corresponding to every  $\varepsilon > 0$ , there is an  $N$  such that

$$\left| f(\mathbf{x}) - \sum_{k=1}^n f_k(\mathbf{x}) \right| < \varepsilon \quad \text{for all } n \geq N \text{ and } \mathbf{x} \in X$$

Since the limit of a uniformly convergent sequence of functions is continuous (Theorem 2.14), we can assert that the sum of a convergent series of continuous functions is continuous. Likewise, from the Cauchy criterion for sequences, we obtain a corresponding criterion for the convergence of series.

**Proposition 1. (Cauchy Criterion)** *Let  $\{f_k\}$  be a sequence of continuous functions. The series  $\sum f_k$  converges if and only if, to each  $\varepsilon > 0$  there correspond an  $N$  such that*

$$\left\| \sum_{k=n+1}^m f_k \right\| < \varepsilon \quad \text{for all } n, m \geq N$$

*Proof.* We must show that the sequence  $g_n = \sum_{k=1}^n f_k$  satisfies the Cauchy criterion. For a given  $\varepsilon > 0$ , let  $N$  be as in the proposition. Then, for any  $\mathbf{x}$ ,  $m > n \geq N$

$$|g_m(\mathbf{x}) - g_n(\mathbf{x})| = \left| \sum_{k=n+1}^m f_k(\mathbf{x}) \right| \leq \left\| \sum_{k=n+1}^m f_k \right\| < \varepsilon$$

Thus  $\|g_m - g_n\| \leq \varepsilon$ , so the proposition is proven.

Notice that the Cauchy criterion is guaranteed if the series of real numbers  $\sum_{k=1}^{\infty} \|f_k\|$  converges (for  $\|\sum_{k=n+1}^m f_k\| \leq \sum_{k=n+1}^m \|f_k\|$ ). This gives us a powerful technique for verifying convergence of series.

**Definition 2.** Let  $\{f_k\}$  be a sequence of continuous functions defined on a set  $X$ . The series is said to **converge absolutely** if  $\sum_{k=1}^{\infty} \|f_k\| < \infty$ .

Of course, as remarked above, an absolutely convergent series is convergent. In the case of absolute convergence we can pose a comparison test, just as for series of numbers.

**Theorem 5.1. (Comparison Test)** *Let  $\{f_k\}$  be a sequence of continuous functions defined on a set  $X$ . Suppose there is a sequence  $\{p_k\}$  of positive numbers and an integer  $N > 0$  such that*

$$(i) \|f_k\| < p_n \quad \text{for } k \geq N$$

$$(ii) \sum_{k=1}^{\infty} p_k < \infty$$

Then  $\sum f_k$  converges absolutely.

*Proof.* The verification is the same as that for number series (Theorem 2.3).

### Examples

1.  $\sum z^k = 1/(1-z)$  uniformly and absolutely in  $\{|z| \leq r\}$  for any  $r < 1$ . For  $\|z^k\| \leq r^k$  in that domain, and

$$\sum r^k = \frac{1}{1-r} < \infty$$

2.  $\sum z^k$  does *not* converge uniformly in  $\{|z| < 1\}$ . In fact, the series is not a Cauchy sequence of functions, because for every  $n$ ,

$$\left\| \sum_{k=1}^{n+1} f_k - \sum_{k=1}^n f_k \right\| = \|z^{n+1}\| = 1$$

Thus, for  $\varepsilon = \frac{1}{2}$ , say, there is no  $N$  such that  $\|\sum_{k=n}^m f_k\| < \frac{1}{2}$  for all  $m > n \geq N$ , in fact, not even for  $m = n + 1$ .

3.  $e^z = \sum_{k=1}^{\infty} z^k/k!$  converges uniformly in any disk  $\{|z| < R\}$  with  $R$  finite. Again, by comparison

$$\left| \frac{z^k}{k!} \right| \leq \frac{R^k}{k!} \quad \text{and} \quad \sum \frac{R^k}{k!} < \infty$$

$$4. \sum \frac{\cos nx}{n^2}$$

converges uniformly on the whole real line. For any  $x$ ,

$$\left| \frac{\cos nx}{n^2} \right| \leq \frac{1}{n^2}$$

Since  $\sum 1/n^2 < \infty$  the comparison test easily applies.

5. If  $\{a_k\}$  is any sequence of numbers such that  $\sum |a_k| < \infty$ , then  $f(z) = \sum_{k=1}^{\infty} a_k z^k$  is a continuous function on the closed unit disk.

The series converges uniformly since  $\|a_k z^k\| \leq |a_k|$ .

Finally, for the purpose of availability, we record the obvious extensions to series of the propositions concerning integration and differentiation of sequences of functions.

**Proposition 2.**

(i) Let  $\{f_n\}$  be a sequence of continuous functions on the interval  $[a, b]$ . Let  $g_n(x) = \int_a^x f_n$ . If the series of functions  $\sum f_n$  converges, so does the series  $\sum g_n$ , and

$$\sum_{n=1}^{\infty} \int_a^x f_n = \int_a^x \left( \sum_{n=1}^{\infty} f_n \right) \quad (5.1)$$

(ii) Let  $\{f_n\}$  be a sequence of continuously differentiable functions on the interval  $[a, b]$ . Let  $g_n = f_n'$ . If the series of functions  $\sum g_n$  converges, and for some  $c$ ,  $\sum f_n(c)$  converges, then the series  $\sum f_n$  converges. The limit is continuously differentiable and

$$\left( \sum f_n \right)' = \sum f_n' \quad (5.2)$$

**Examples**

6.

$$\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \quad -1 < x < 1$$

This follows by integrating the geometric series (Example 1) term by term:

$$\int_0^x \frac{1}{1-t} dt = \sum_{k=0}^{\infty} \int_0^x t^k dt$$

$$\ln(1-x) = \sum_{k=0}^{\infty} \frac{t^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{t^k}{k}$$

7.

$$f(x) = \sum_{k=1}^{\infty} \frac{\cos nx}{n!}$$

is infinitely differentiable. For the differentiated series

$$- \sum_{n=1}^{\infty} \frac{\sin nx}{(n-1)!} \tag{5.3}$$

is also convergent. By Proposition 2(ii) the sum is  $f'(x)$ . Similarly, the series (5.3) can be differentiated term by term, and gives

$$- \sum_{n=1}^{\infty} \frac{n \cos nx}{(n-1)!}$$

which is again convergent.

8. We can develop a series expansion for arc tan  $x$  according to the following observations. From the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

we obtain by substituting  $-x^2$  for  $x$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

Integrate:

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

● EXERCISES

1. For what values of  $x$  do these series of functions converge absolutely:

(a)  $\sum_{n=0}^{\infty} 2^n x^n$       (d)  $\sum_{n=0}^{\infty} (x+18)^n$

(b)  $\sum_{n=1}^{\infty} \frac{\cos nx}{x^n}$       (e)  $\sum_{n=0}^{\infty} e^{\sin nx}$

(c)  $\sum_{n=0}^{\infty} e^{nx}$       (f)  $\sum_{n=0}^{\infty} x^{(n^2)}$

2. In which domains of the complex plane do these series converge?

(a)  $\sum_{n=0}^{\infty} nz^n$       (b)  $\sum_{n=0}^{\infty} \frac{z^n}{(2n)!}$       (c)  $\sum_{n=0}^{\infty} n^{-z} e^{inz}$

3. Which of these series can be differentiated or integrated on their domain of convergence?

(a) Exercise 1(a)      (c) Exercise 1(d)

(b) Exercise 1(b)      (d)  $\sum_{n=0}^{\infty} \frac{\cos nx}{n^2}$

4. Find the power series expansion for these functions:

(a)  $\frac{1}{(1+x^2)^2}$       (c)  $\int_0^x e^{t^2} dt$

(b)  $\frac{d}{dx} \left( \frac{1+x}{1-x} \right)$       (d)  $\int_0^x \frac{dt}{1+t^4}$

● PROBLEMS

- (a) Find a power series expansion for  $\sin x \cos x$ .  
(Hint:  $2 \sin x \cos x = \sin 2x$ .)  
(b) Find power series expansions for  $\sin^2 x$  and  $\cos^2 x$ .
- Prove Proposition 2.
- Show that

$$\lim_{x \rightarrow -1} \sum_{n=1}^{\infty} \frac{x^n}{n} = \log 2$$

Can you conclude

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n} = \log 2?$$

## 5.2 The Fundamental Theorem of Algebra

For the remainder of this chapter we restrict attention exclusively to complex-valued functions of a complex variable. The simplest class of such functions are the **polynomial functions**; that is, functions of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

(where the  $a_i$  are complex numbers). We shall always assume  $a_n \neq 0$ ; in this case  $n$  is called the **degree of  $P$** . It is a basic fact of mathematics that every polynomial has a root; that is, there is a number  $c \in C$  such that  $P(c) = 0$ . The proof of this fact consists in a systematic investigation of the analytic properties of polynomials. First, we recall de Moivre's theorem.

**Lemma.** *Every nonzero complex number has  $n$  distinct  $n$ th roots.*

*Proof.* Let  $c \in C$ ,  $c \neq 0$ . For this purpose, the polar representation  $c = r e^{i\theta}$  is most convenient. An  $n$ th root of  $c$  is a number  $w = \rho e^{i\phi}$  such that  $\rho^n = r$  and  $e^{in\phi} = e^{i\theta}$ ; that is,  $n\phi - \theta$  is an integral multiple of  $2\pi$ . Let

$$\alpha_1 = \frac{2\pi}{n}, \alpha_2 = \frac{4\pi}{n}, \dots, \alpha_k = \frac{2\pi k}{n}, \dots, \alpha_{n-1} = \frac{2\pi(n-1)}{n}, \alpha_n = 2\pi$$

Then  $\omega_1 = \exp(i\alpha_1), \dots, \omega_n = \exp(i\alpha_n)$  are all distinct and have the property  $(\omega_n)^n = 1$ . These are called the  **$n$ th roots of unity**. Now, if  $\rho = (r)^{1/n}$  and  $\phi = \theta/n$ , then  $(\rho e^{i\phi})^n = r e^{i\theta} = c$ . The numbers  $\rho e^{i\phi}\omega_1, \dots, \rho e^{i\phi}\omega_k, \dots, \rho e^{i\phi}\omega_n$  are then all distinct, and are all  $n$ th roots of  $c$ .

Now, we need two deeper facts depending on the continuity properties of polynomials. The first is intuitively clear: that  $|P(z)|$  gets arbitrarily large as  $z \rightarrow \infty$ . The second is the crucial fact for the fundamental theorem: the place where a polynomial has a minimum modulus must be a root.

**Lemma.** *Let  $P(z) = a_n z^n + \cdots + a_1 z + a_0$  be a polynomial of degree  $n > 0$ .*

(i)  $\lim_{|z| \rightarrow \infty} |P(z)| = \infty$ , that is, given any  $M > 0$  there is a  $K > 0$  such that  $|P(z)| \geq M$  whenever  $|z| \geq K$ .

(ii) If  $P(z_0) \neq 0$ , then  $z_0$  cannot be a minimum point for  $|P|$ ; that is, there are  $z$  close to  $z_0$  such that  $|P(z)| < |P(z_0)|$ .

*Proof.*

(i) The point here is that the highest-degree term of  $P$  is the dominating term as regards the behavior of  $P$  as  $z \rightarrow \infty$ . For  $z \neq 0$ ,

$$|P(z)| = |z|^n \left| a_n + \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}} \right|$$

If  $|z| \geq K \geq 1$ , then  $|z|^{n-k} \geq K$  also for  $k < n$ , so

$$\left| a_n + \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}} \right| \geq |a_n| - \frac{1}{K} \left( \sum_{k=0}^{n-1} |a_k| \right)$$

Let  $M > 0$  be given, and choose

$$K = \max \left( 1, 2M |a_n|^{-1}, 2 |a_n|^{-1} \left( \sum_{k=0}^{n-1} |a_k| \right) \right)$$

Then, for  $|z| \geq K$ ,

$$\left| a_n + \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}} \right| \geq |a_n| \left( 1 - \frac{|a_n|^{-1} \left( \sum_{k=0}^{n-1} |a_k| \right)}{K} \right) \geq \frac{1}{2} |a_n|$$

Thus

$$|P(z)| \geq |z|^n \cdot \frac{1}{2} |a_n| \geq K \cdot \frac{1}{2} |a_n| \geq M$$

(ii) Suppose now that  $P(z_0) \neq 0$ . Let

$$Q(z) = (P(z_0))^{-1} P(z + z_0)$$

Then  $Q$  is also a polynomial,  $Q(0) = 1$ , and we must show that 0 is not a minimum point for  $Q$ . Let

$$Q(z) = 1 + \sum_{k=1}^n a_k z^k = 1 + z^m (a_m + z g(z))$$

where  $m$  is chosen as the least positive integer  $k$  for which  $a_k \neq 0$ , and  $g(z) = \sum_{k=m+1}^n a_k z^{k-(m+1)}$ .  $g$  is a polynomial and is thus continuous (and that is all we need to know about  $g$ ). Here again we want to use the fact that for small  $z$ ,  $z^m$  dominates  $z^{m+1}$ , so  $Q$  is very close to the polynomial  $1 + z^m a_m$  which has no minimum modulus at 0 (choose  $z$  so that  $z^m = -r/a_m$  with  $r < 1$ ).

In our case, we choose an  $m$ th root of  $-a_m^{-1}$ ; call it  $z_0$ , and consider the function



$Q(rz_0)$  of a real variable  $r$ . We have

$$Q(rz_0) = 1 + r^m(-1 + rh(r))$$

where  $h(r) = -a_m^{-1}g(rz_0)$  is a continuous complex-valued function. Thus

$$|Q(rz_0)| \leq |1 - r^m| + r^{m+1} |h(r)|$$

Now  $\lim_{r \rightarrow 0} rh(r) = 0$ , so we can choose  $r_0 < 1$  small enough so that  $|r_0 h(r_0)| \leq \frac{1}{2}$ .

Then

$$|Q(rz_0)| \leq 1 - r_0^m + r_0^m(\frac{1}{2}) \leq 1 - \frac{1}{2}r_0^m < 1$$

which proves part (ii).

**Theorem 5.2. (Fundamental Theorem of Algebra)** *Let  $P$  be a polynomial of positive degree. There is a  $z_0 \in C$  such that  $P(z_0) = 0$ .*

*Proof.* Let  $P(0) = c_0$ . By part (i) of the lemma, there is a  $K > 0$  such that for  $|z| \geq K$ ,  $|P(z)| \geq |c_0|$ . Now  $\Delta = \{z \in C; |z| \leq K\}$  is compact, so  $|P|$  attains a minimum value on  $\Delta$ , say at  $z_0$ . But then  $z_0$  is a minimum point for all of  $C$ . For, since  $0 \in \Delta$ ,  $|P(z_0)| \leq |P(0)| = c_0$ , and for  $z \notin \Delta$ ,  $|P(z)| \geq c_0 \geq |P(z_0)|$ . Thus, even for  $z \notin \Delta$ , we have  $|P(z_0)| \leq |P(z)|$ . But then, by part (ii), there is no alternative: we must have  $P(z_0) = 0$ .

### Factorization Theory

We should recall that if  $c$  is a zero of the polynomial  $P$ , then  $z - c$  factors  $P$  (this is proven below in Theorem 5.3). Thus  $P(z) = (z - c)Q(z)$  and  $Q$  has degree 1 less than that of  $P$ . If  $\deg Q > 0$ ,  $Q$  has a zero  $c'$ , which is also a zero of  $P$ . Further,  $Q(z) = (z - c')Q'(z)$  and we can repeat this argument in order to find exactly  $\deg P$  zeros of  $P$ . This is the factorization theorem of algebra.

**Theorem 5.3. (Factorization Theorem)** *Let  $P$  be a polynomial of degree  $n > 0$ . There are complex numbers  $a \neq 0, z_1, \dots, z_n$  such that*

$$P(z) = a(z - z_1) \cdots (z - z_n)$$

*Proof.* The proof is by induction on  $n$ . If  $n = 1$  the situation is simple:

$$P(z) = a_1z + a_0 = a_1 \left( z - \left( \frac{-a_0}{a_1} \right) \right)$$

(since  $a_1 \neq 0$ ). Now we consider the case of general degree  $n$ , assuming the corollary for polynomials of degree  $n-1$ . By the theorem, there is a point  $c$  such that  $P(c) = 0$ . Then

$$\begin{aligned} P(z) - P(c) &= \sum_{k=1}^n a_k(z^k - c^k) = \sum_{k=1}^n a_k(z-c) \left( \sum_{j=0}^{k-1} z^j c^{k-1-j} \right) \\ &= (z-c) \sum_{j=0}^{n-1} \left( \sum_{k=j+1}^n a_k c^{k-1-j} \right) z^j \end{aligned}$$

The factor on the right is a polynomial of degree  $n-1$ , so the induction assumption applies: it can be written as  $a(z-z_1) \cdots (z-z_{n-1})$  for suitable  $a \neq 0$ ,  $z_1, \dots, z_{n-1}$ . Thus, writing  $c = z_n$ , we obtain

$$P(z) = a(z-z_1) \cdots (z-z_n) \quad (5.4)$$

This factorization is clearly unique, except for the order of the  $z_i$ 's:  $a$  is the leading coefficient of  $P$  and  $\{z_1, \dots, z_n\}$  are the roots of  $P$ . Of course,  $z_1, \dots, z_n$  need not be distinct; let  $r_1, \dots, r_s$  be the set of distinct roots. If we let  $m_i$  be the number of occurrences of the root  $r_i$  in the list  $\{z_1, \dots, z_s\}$ ,  $m_i$  is called the **multiplicity** of the root  $r_i$ . We can rewrite (5.4) as

$$P(z) = a(z-r_1)^{m_1} \cdots (z-r_s)^{m_s} \quad (5.5)$$

and clearly  $m_1 + \cdots + m_s = n$ , the degree of  $P$ .

Before concluding this section we should remark on the factorization of real polynomials. Real polynomials need not have real roots (viz.,  $z^2 + 1 = 0$ ), but their complex roots come in conjugate pairs. Let  $P(z) = a_n z^n + \cdots + a_1 z + a_0$  be a real polynomial. If  $P(r) = 0$ , then

$$P(\bar{r}) = a_n(\bar{r})^n + \cdots + a_1 \bar{r} + a_0 = (a_n r^n + \cdots + a_1 z + a_0)^- = P(r)^- = 0$$

so  $\bar{r}$  is also a root of  $P$ . Since

$$(z-r)(z-\bar{r}) = z^2 - (r+\bar{r})z + r\bar{r} = z^2 - 2\operatorname{Re}(r)z + |r|^2$$

the polynomial has real coefficients. Thus, if we rearrange the roots of  $P$  into the real roots  $r_1, \dots, r_k$  and the conjugate pairs  $r_{k+1}, \bar{r}_{k+1}, \dots, r_t, \bar{r}_t$  we can rewrite the factorization (5.5) into a product of linear and quadratic real polynomials.

$$P(z) = a(z-r_1)^{m_1} \cdots (z-r_k)^{m_k} (z^2 - 2\operatorname{Re}(r_{k+1})z + |r_{k+1}|^2) \cdots (z^2 - 2\operatorname{Re}(r_t)z + |r_t|^2)$$

## ● PROBLEMS

4. Let  $\omega_1, \dots, \omega_n$  be the  $n$ th roots of unity. Show that they are arranged at  $n$  equidistant points around the unit circle. Show that the sets  $\{\omega_1, \dots, \omega_n\}$ ,  $\{\omega_1, \omega_1^2, \dots, \omega_1^{n-1}\}$  are the same, if  $\omega_1$  is the nearest such point to 1.

5. Let  $\omega_1, \dots, \omega_n$  be the  $n$ th roots of unity. Choose  $k$  so that  $kn - 2$  is divisible by 4. Show that  $i^k \omega_1, \dots, i^k \omega_n$  are the  $n$ th roots of  $-1$ .

6. Show that: (a)  $\deg PQ = \deg P + \deg Q$ .

(b)  $\deg(P + Q) = \max(\deg P, \deg Q)$  if  $\deg P \neq \deg Q$ .

(c) When is the equation in (b) not true?

7. Given two polynomials  $P, Q$  show that there is a polynomial  $R$  which factors both  $P, Q$  and is factored by any polynomial which factors both  $P, Q$ .  $R$  is called the **greatest common divisor of  $P$  and  $Q$** .

8. Show that a real polynomial of odd degree has a real root.

9. Prove that the polynomial  $1 + z^m a_m$  ( $m > 0$ ) has no minimum modulus at  $z = 0$ .

10. For  $P(z) = \sum_{n=0}^N a_n z^n$  a polynomial, let

$$P'(z) = \sum_{n=1}^N n a_n z^{n-1}$$

(a) Verify that the transformation  $P \rightarrow P'$  is linear and satisfies

$$(PQ)' = PQ' + P'Q$$

(by induction on  $\deg P$ ).

( $P \rightarrow P'$  is a complex analog of differentiation)

(b) Prove that  $r$  is a multiple root of  $P$  if and only if  $P(r) = 0$  and  $P'(r) = 0$ .

(c) Define  $P'' = (P')'$ ,  $P''' = (P'')'$ , and so on. Then  $r$  is a root of  $P$  of at least multiplicity  $m$  if and only if  $P(r) = P'(r) = \dots = P^{(m-1)}(r) = 0$ .

### 5.3 Constant Coefficient Linear Differential Equations

Now that we know the factorization theorem for polynomials we can return to complete the study of constant coefficient equations in one unknown function. Let  $L$  be a **constant coefficient differential operator of order  $k$** ; that is,  $L$  is a mapping from functions to functions defined by

$$L(f) = f^{(k)} + \sum_{i=0}^{k-1} a_i f^{(i)} \quad a_i \in C \quad (5.6)$$

Corresponding to  $L$  is the polynomial

$$P_L(X) = X^k + \sum_{i=0}^{k-1} a_i X^i$$

called **the characteristic polynomial of  $L$** . We recall the facts that we already know about such differential operators.

**Theorem 5.4.** *Let  $L$  be given by (5.6). The collection  $S(L)$  of solutions of the equation  $Lf = 0$  is an  $n$ -dimensional vector space of infinitely differentiable functions. If  $r$  is a root of  $P_L(X) = 0$ , then  $e^{rx} \in S(L)$ .*

Now if all the roots of the characteristic polynomial are distinct, we have  $n$  solutions of  $Lf = 0$ , and it is easily verified (Problem 11) that they are independent. Thus they span  $S(L)$ . To examine the case of multiple roots, we must examine more closely the relationship between the given differential operator and its characteristic polynomial. If  $P$  is a polynomial, we will let  $L_P$  represent the corresponding operator; that is, for  $P(X) = \sum_{i=0}^n a_i x^i$ ,  $L_P$  is defined by

$$L_P(f) = \sum_{i=0}^n a_i f^{(i)}$$

Now, from what we already know about these differential equations we can guess that the factorization of  $P$  will tell us all we want to know about  $L_P$ . In fact, we can factor the corresponding operator accordingly as the next lemma shows.

**Lemma 1.**  $L_{P+Q} = L_P + L_Q$ ;  $L_{PQ} = L_P L_Q$ .

*Proof.* Of course,  $L_P L_Q$  is defined as the composition of operators:  $(L_P L_Q)(f) = L_P(L_Q(f))$ . The first equation is obvious. The second takes a little work. We will prove it by induction on the degree of  $P$ . If  $\deg P = 0$ , that is,  $P(x) = a_0$ , then  $PQ = a_0 Q$  and  $L_{PQ}(f) = a_0 L_Q(f) = L_P(L_Q(f))$ , for any sufficiently differentiable function  $f$ . Now suppose the lemma is true for all polynomials of degree  $n$ . Let  $P$  be a polynomial of degree  $n+1$ . If  $a$  is a root of  $P$ , we can write  $P(X) = (X-a)S(X)$ , where  $S$  is a polynomial of degree  $n$ . Thus, by hypothesis,  $L_{SQ} = L_S L_Q$ . We have left only to verify the lemma for polynomials of degree 1. That is, we must show that if  $R$  is a polynomial of degree 1 and  $T$  is any polynomial, then  $L_{RT} = L_R L_T$ . For once this is verified, we take  $R(X) = X-a$ , so that  $P = RS$ . Then

$$L_{PQ} = L_{RSQ} = L_R L_{SQ} = L_R L_S L_Q = L_{RS} L_Q = L_P L_Q$$

So, let  $R(X) = X - a$ ,  $T(X) = \sum_{i=0}^m b_i X^i$ . Then

$$RT(x) = \sum_{i=0}^m (b_i - ab_{i+1})X^{i+1} - ab_0$$

Now we compute  $L_R L_T$ :

$$\begin{aligned} L_R L_T(f) &= L_R \left( \sum_{i=0}^m b_i f^{(i)} \right) = \sum_{i=0}^m (b_i f^{(i)})' - a \sum_{i=0}^m b_i f^{(i)} \\ &= \sum_{i=0}^m (b_i - ab_{i+1}) f^{(i+1)} - ab_0 f \end{aligned}$$

The lemma is proven.

It follows from the lemma that if  $Q$  is a factor of  $P$ , then any solution of  $L_{PQ}(f) = 0$  is a solution of  $L_P(f) = 0$ . Now let  $P$  be a given polynomial. We can, by the factorization theorem, write  $P$  as a product of first-order factors.

$$P(X) = (X - a_1)^{m_1} \cdots (X - a_s)^{m_s} \quad \text{with } m_1 + \cdots + m_s = \deg P$$

Because of Lemma 1 the solutions corresponding to the factors  $(X - a_j)^{m_j}$  are in  $S(L_P)$ . Thus we need to discover the solutions of the differential equation  $L_P(f) = 0$ , where  $P(X) = (X - c)^m$ .

Consider, for example, the differential operator corresponding to  $(X - c)^2$ . We know one solution:  $e^{cx}$ ; we find another by the technique of variation of parameters.  $(X - c)^2 = X^2 - 2cX + c^2$ . Test the operator on  $y = ze^{cx}$ .

$$y' = z'e^{cx} + zce^{cx} \quad y'' = z''e^{cx} + 2z'ce^{cx} + zc^2e^{cx}$$

Then

$$y'' - 2cy' + c^2y = z''e^{cx} = 0 \quad \text{or } z'' = 0$$

Thus  $z = x$ , and the second solution is  $xe^{cx}$ . We can guess then that the general situation is this.

**Lemma 2.** *The solutions of  $L_{(X-c)^m}(f) = 0$  are spanned by  $e^{cx}$ ,  $xe^{cx}$ ,  $\dots$ ,  $x^{m-1}e^{cx}$ .*

*Proof.* We have to show that the named functions are solutions. We do that by induction. The case  $m = 1$  is already known (by Lemma 1). Thus we may

assume the lemma for a given value of  $m$ , and prove it for  $m + 1$ . By Lemma 2, we need only verify that  $L_{(X-c)^{m+1}}(x^m e^{cx})$  is zero. But this is

$$\begin{aligned} L_{(X-c)^m} L_{(X-c)}(x^m e^{cx}) &= L_{(X-c)^m}(mx^{m-1}e^{cx} + cx^m e^{cx} - cx^m e^{cx}) \\ &= L_{(X-c)^m}(mx^{m-1}e^{cx}) = 0 \end{aligned}$$

by induction.

**Theorem 5.5.** Let  $p(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$  be a polynomial with complex coefficients. Let  $a_1, \dots, a_s$  be the roots of  $p(X) = 0$  with multiplicities  $m_1, \dots, m_s$ , respectively. Then the space  $S(L_p)$  of solutions of the differential equation

$$L_p(f) = f^{(n)} + \sum_{i=0}^{n-1} a_i f^{(i)} = 0$$

is the linear span of the functions  $x^j e^{a_i x}$ ,  $0 \leq j < m_i$ .

### ● EXERCISES

5. Solve these differential equations:

- $y''' - 5y'' + 8y' - 4y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 0$ ,  $y''(0) = 1$ .
- $y'' - y' - 5y' - 3y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 2$ ,  $y''(0) = -1$ .
- $y''' - 6y'' + 12y' - 8y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$ ,  $y''(0) = 1$ .
- $y''' - 3y'' + 3y' - y = 0$ ,  $y(0) = 3$ ,  $y'(0) = 2$ ,  $y''(0) = 1$ .
- $y^{(4)} + 2y'' + y = 0$ ,  $y(0) = 2$ ,  $y'(0) = 2$ ,  $y''(0) = 2$ ,  $y'''(0) = 2$ .
- $y^{(4)} + 4y^{(3)} - 2y^{(2)} - 12y' + 9y = 0$ ,  $y(0) = y'(0) = y''(0) = 1$ ,  $y'''(0) = 0$ .
- $y^{(4)} - 3y^{(3)} + 2y = 0$ ,  $y(0) = 0$ ,  $y'(0) = y''(0) = y'''(0) = 1$ .

### ● PROBLEMS

11. (a) Show that if  $r_1, \dots, r_n$  are  $n$  distinct numbers, the matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ r_1 & \cdots & r_n \\ r_1^2 & \cdots & r_n^2 \\ \vdots & \cdots & \vdots \\ r_1^{n-1} & \cdots & r_n^{n-1} \end{pmatrix}$$

is nonsingular. (*Hint:* If the rows are dependent, we obtain a polynomial of degree  $n - 1$  with  $n$  distinct roots.)

(b) The functions  $\exp(r_1 x), \dots, \exp(r_n x)$  are independent. (*Hint:* If these functions were dependent, we would be able to prove that the columns of the above matrix are dependent.)

### 5.4 Solutions in Series

If now we are given a linear differential equation which is not homogeneous, or has variable coefficients, we have a problem of a much different magnitude. In general, such problems cannot be solved explicitly. Thus, we must seek ways to obtain approximate solutions. This is one of the places where series representations of functions are usable. The procedure of series approximation has two aspects. First, we must establish the theoretical validity of such a technique and, secondly (and this is essential from the practical point of view), we need a technique for effectively computing the error. In this section we shall describe this procedure, deferring these two essential points (which turn out to be the same!) until Section 5.7.

First, an example. Suppose we want the function  $f$  such that

$$f''(x) + g_1(x)f'(x) + g_0(x)f(x) = 0 \quad f(0) = a_0 \quad f'(0) = a_1$$

where  $g_0$  and  $g_1$  are defined in a neighborhood of 0. We shall assume that they are sufficiently differentiable. Now our initial conditions give us the first two terms of the Taylor expansion of  $f$  at 0:

$$f(x) = a_0 + a_1x + \text{higher-order terms} \quad (5.7)$$

Our technique will be based on the tacit assumption that the “higher-order terms” are computable, and knowing enough of them will give a usable approximation to the solution. Now, evaluating the differential equation itself at 0 gives us the second-order term:

$$f''(0) + g_1(0)f'(0) + g_0(0)f(0) = 0$$

or

$$f''(0) = -(g_1(0)a_1 + g_0(0)a_0)$$

so

$$f(x) = a_0 + a_1x - \frac{1}{2}(a_0g_0(0) + a_1g_1(0))x^2 + \text{higher-order terms} \quad (5.8)$$

Differentiating the differential equation will give an identity express-

ing  $f'''$  in terms of lower derivatives, so we may continue,

$$f'''(x) + g_1'(x)f'(x) + g_1(x)f''(x) + g_0'(x)f(x) + g_0(x)f'(x) = 0$$

so

$$\begin{aligned} f'''(0) &= -(g_1'(0) + g_0(0))a_1 - g_0'(0)a_0 + g_1(0)(g_1(0)a_1 + g_0(0)a_0) \\ &= (g_1(0)^2 - g_1'(0) - g_0(0))a_1 + (g_0(0)g_1(0) - g_0'(0))a_0 \end{aligned}$$

and so we have the third term of the Taylor series of  $f$ :

$$\begin{aligned} f(x) &= a_0 + a_1x - \frac{1}{2}(a_0g_0(0) + a_1g_1(0))x^2 \\ &\quad + \frac{1}{6}[(g_1(0)^2 - g_1'(0) - g_0(0))a_1 + g_0(0)g_1(0) - g_0'(0)]x^3 \\ &\quad + \text{higher-order terms} \end{aligned}$$

### Example

9. Perhaps an explicit calculation is in order. We shall find an approximate solution of:

$$\begin{aligned} y'' + (x^2 - 1)y' + xy &= x^2 \\ y(0) = 0 \quad y'(0) &= 2 \end{aligned} \tag{5.9}$$

The solution thus begins  $f(x) = 2x + \dots$ .  $f''(0)$  is easy to calculate by substituting the initial conditions into Equation (5.9):

$$f(x) = 2x + x^2 + \dots$$

Differentiating (5.9), we obtain

$$y''' + 2xy' + (x^2 - 1)y'' + y + xy' - 2x \tag{5.10}$$

Evaluating at 0 we find  $f'''(0) = f''(0) - f(0) = 2$ . Differentiating (5.10) and evaluating at 0, we obtain  $f^{(4)}(0) = 2$ ; once again gives  $f^{(5)}(0) = -10$ . Thus, to five terms the Taylor expansion of the desired solution is

$$f(x) = 2x + x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{1}{12}x^5 + \text{higher-order terms}$$

Admittedly this is not very glamorous, but it is computable! The phrase



“higher-order terms” represents the error between the fifth-degree polynomial exhibited above and the actual solution. That polynomial is completely meaningless without some estimate on the error incurred. But our method gives no hint as how to estimate. So, in the hope of being able to give more form to the “higher-order terms,” we will try a more brazen approach: we begin by assuming that the desired solution is the sum of a convergent power series (its “full Taylor expansion”) and we try to find the coefficients. If  $f(x) = \sum_{n=0}^{\infty} a_n x^n$ , then differentiating term by term we obtain

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n x^{n-k}$$

We make these substitutions into the given differential equations and solve for the  $\{a_n\}$  by equating the coefficients of  $f^{(k)}$ .

Let us reconsider (5.9). Let  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  be the desired solution. The statement of the problem becomes

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + (x^2 - 1) \sum_{n=1}^{\infty} a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n - z^2 = 0 \quad (5.11)$$

$$a_0 = 0 \quad a_1 = 2$$

The coefficient of  $x^k$  in the left-hand side of (5.11) is

$$(k+2)(k+1)a_{k+2} + (k-1)a_{k-1} - (k+1)a_{k+1} + a_{k-1}$$

Thus we have to solve these equations

$$\begin{array}{ll} a_0 = 0 & \\ a_1 = 2 & \text{(initial conditions)} \\ 2a_2 - a_1 = 2 & (k=0) \\ 3.2a_3 + a_0 - 2a_2 = 0 & (k=1) \\ 4.3a_4 + 2a_1 - 3a_3 = 0 & (k=2) \\ \vdots & \\ n(n-1)a_n + (n-2)a_{n-3} - (n-1)a_{n-1} = 0 & (k=n-2) \\ \vdots & \end{array}$$

We can solve, because each equation can be written in the form

$$a_n = \frac{(n-1)a_{n-1} - (n-2)a_{n-3}}{n(n-1)} \quad n > 2 \quad (5.12)$$

However, we have an added advantage in that we can make a guess at an estimate for the general term  $a_n$ . In fact, we assert

$$|a_n| \leq \frac{2^n}{[n/3]!} \quad (5.13)$$

( $[x]$  = largest integer less than or equal to  $x$ ). This is in fact true for  $n = 0, 1, 2$ ; we verify it in general by induction.

$$\begin{aligned} |a_n| &\leq \frac{(n-1)|a_{n-1}| + (n-2)|a_{n-3}|}{n(n-1)} \\ &\leq \frac{1}{n} (|a_{n-1}| + |a_{n-3}|) \\ &\leq \frac{1}{n} \left( \frac{2^{n-1}}{[(n-1)/3]!} + \frac{2^{n-3}}{[(n-3)/3]!} \right) \end{aligned}$$

Now, since

$$\frac{(n-3)}{3} = \frac{n}{3} - 1 \quad \left[ \frac{n}{3} \right] \left[ \frac{(n-3)}{3} \right]! = \left[ \frac{n}{3} \right]!$$

so

$$\left( \frac{n}{3} \right) \left[ \frac{(n-3)}{3} \right]! \geq \left[ \frac{n}{3} \right]!$$

Similarly,

$$\left( \frac{n}{3} \right) \left[ \frac{(n-1)}{3} \right]! \geq \left[ \frac{n}{3} \right]!$$

Thus,

$$|a_n| \leq \frac{1}{3[n/3]!} (2^{n-1} + 2^{n-3}) \leq \frac{2^n}{[n/3]!}$$

This now tells us a lot. For, the solution to the problem in (5.9) differs from

$$2x + x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4 - \frac{1}{12}x^5$$

by at most  $\sum_{n \geq 6} a_n x^n$ , where the  $a_n$  satisfy (5.13). Thus the error is dominated by

$$\begin{aligned} \sum_{n \geq 6} \frac{2^n}{[n/3]!} |x|^n &= \sum_{k \geq 2} \frac{2^{3k} |x|^{3k}}{k!} + \sum_{k \geq 2} \frac{2^{3k+1} |x|^{3k+1}}{k!} + \sum_{k \geq 2} \frac{2^{3k+2} |x|^{3k+2}}{k!} \\ &\leq (1 + 2|x| + 4|x|^2) \cdot (\exp(2|x|)^3 - 1 - (2|x|)^3) \end{aligned}$$

Hence in the interval  $0 \leq x \leq 1$  the solution is given by the above polynomial except for our error of at most  $7e^2$  (which is about 52)! The reader is forgiven if he is unimpressed with our estimate, but he should not go so far as to discard the technique for this reason. For the paucity of our results is due to laziness rather than the uselessness of the Taylor development. If we pushed this procedure up to 1000 terms (an easy task for a computer), then the error would be at most  $7e^2 \cdot 2^{1000}/1000!$  which is less than  $(50)^{-900}$ ; a good estimate indeed.

Let us recapitulate the basic ideas. We are given an initial value problem:

$$y^{(k)} + \sum_{i=0}^{k-1} g_i(x)y^{(i)} = h(x), \quad y(0) = c_0, \quad y'(0) = c_1, \dots, \quad y^{(k-1)}(0) = c_{k-1}$$

We replace the  $g$ 's and  $h$  by power series expansions and test the "solution"  $f(x) = \sum_{k=0}^{\infty} a_n x^n$ . The first  $k$  terms are found from the initial conditions, and the rest are found by equating the coefficient of  $x^n$  on both sides of the equation. This leaves us with these problems to resolve:

- (i) Can we represent the given  $g$ 's and  $h$  by power series?
- (ii) Can we differentiate a power series term by term?
- (iii) How do we multiply power series? (In the above illustration, the  $g$ 's were polynomials, so there was not much difficulty.)
- (iv) Can the system of relations between the  $a_k$ 's really be solved uniquely?
- (v) Can we effectively estimate the error between the solution and a finite part of its (supposed) Taylor expansion?

Little by little, we will resolve these problems. Suffice it to say that the answer to (i) in general is *No* (see Section 5.8). However, in problems that

do arise naturally, the given functions usually are sums of convergent power series. If this is the case, all other questions can be satisfactorily answered; that is, the solution also is the sum of a convergent power series whose coefficients can be determined by the above technique and the estimate on the remainder can be effectively computed. Let us look at another illustration.

### Examples

10.

$$x^3 y''' + x^2 y'' + xy' + y = e^x \quad (5.14)$$

$$y(0) = 1 \quad y'(0) = 1 \quad y''(0) = 1/6$$

Let the solution be  $f(x) = \sum_{n=0}^{\infty} a_n x^n$ . Substituting in (5.14), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} n(n-1)(n-2)a_n x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ = \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

which gives these equations for the coefficients:

$$a_n(n(n-1)(n-2) + n(n-1) + 1) = \frac{1}{n!} \quad \text{for all } n$$

or

$$a_n = \frac{1}{n!(n^3 - 2n^2 - n + 1)} \quad (5.15)$$

Notice, that we have not used the initial conditions and fortunately they conform to the requirements (5.15). That is, for this particular equation, there is a unique solution independent of any initial conditions. This does not contradict any previous results because Picard's theorems do not apply (since the leading coefficient is *not* invertible).

11.

$$y'' - xy' + 2y = 0 \quad (5.16)$$

$$y(0) = 1 \quad y'(0) = 0$$

Here Picard's theorem does apply, so we should get a unique solution with the given initial conditions. Let  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  be the candidate. (5.16) becomes

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

or

$$a_0 = 1 \quad a_1 = 0$$

$$(n+2)(n+1)a_{n+2} - na_n + 2a_n = 0 \quad n \geq 0$$

or

$$a_{n+2} = \frac{(n-2)a_n}{(n+2)(n+1)}$$

Thus  $a_2 = -1$ ,  $a_3 = 0$ ,  $a_4 = 0$  and thus all further coefficients are zero. The solution is  $f(x) = 1 - x^2$ .

In the next section, we shall fully develop the theory of power series. It is most advantageous (as we have already seen) to do so in the complex domain.

### ● EXERCISES

6. Find an approximate solution for

$$y' - xy = 0 \quad y(0) = 0 \quad y'(0) = 1$$

with an error of at most  $10^{-4}$  in the interval  $[-1, 1]$ .

7. Do the same for

$$y'' - x^2y = 1 \quad y(0) = 0 \quad y'(0) = 0 \quad y''(0) = 0$$

with an error of  $10^{-3}$  in  $[-\frac{1}{2}, \frac{1}{2}]$ .

8. Find a recursive formula for the coefficients of the solution, and a reasonable estimate:

(a)  $y'' - 2y' + y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$ .

(b)  $y'' - 2y' + xy = e^x$ ,  $y(0) = 1$ ,  $y'(0) = 1$ .

(c)  $y^{(k)} + y = 1$ , arbitrary initial conditions.

(d)  $y'' - k^2y = 0$ .

(e)  $y' = x^2 + xy$ ,  $y(0) = 0$ .

● PROBLEMS

12. Why doesn't Picard's theorem apply to Equation (5.14)?
13. The second-order equation  $xy'' + y' = 0$  seems to have only one solution by the series method, but two independent solutions by the method of separation of variables. Explain that.

## 5.5 Power Series

We have already discussed at length the power series expansion of the exponential and trigonometric functions, the geometric series and some others. We have also seen that the Taylor formula produces a power series expansion for suitable functions. We have observed that there is a certain disk corresponding to each power series, called the disk of convergence. The series converges inside that disk and diverges outside. We shall recollect all this information as the starting point of our discussion of complex power series.

**Theorem 5.6.** *Let  $c_n$  be a sequence of complex numbers. There is a non-negative number  $R$  (called the **radius of convergence** of the power series  $\sum c_n z^n$ ) with these properties:*

- (a)  $\sum_{n=0}^{\infty} c_n z^n$  diverges if  $|z| > R$ .
- (b)  $\sum_{n=0}^{\infty} c_n z^n$  converges absolutely and uniformly in any disk  $\{z \in \mathbb{C}: |z| \leq r\}$  with  $r < R$ .

$R$  has these two descriptions:

- (i)  $R = \text{l.u.b. } \{t: |c_n|t^n \text{ is bounded}\}$ .
- (ii)  $R = (\limsup(|c_n|)^{1/n})^{-1}$ .

*Proof.* For at least part of the proof we could refer to Proposition 9. As in that proposition we consider the set

$$\{t \geq 0: \text{there is an } M \text{ such that } M \geq |c_n|t^n \text{ for all } n\}$$

If this set is unbounded, we can take  $R = \infty$ , otherwise, let  $R$  be the least upper bound of this set.

(a) Suppose  $|z| \geq R$ . Then there is a  $t$ ,  $|z| > t \geq R$  such that  $\{|c_n|t^n\}$  is unbounded. Since  $|c_n||z|^n \geq |c_n|t^n$  for all  $n$ , we cannot have  $\lim c_n z^n = 0$  so  $\sum c_n z^n$  diverges.

(b) Let  $r < R$ ,  $\Delta = \{z \in \mathbb{C}: |z| \leq r\}$ . Then there is a  $t$ ,  $r < t \leq R$  such that

$\{|c_n|t^n\}$  is bounded, say by  $M$ . If  $|z| \leq r$ ,

$$|c_n z^n| = |c_n| t^n \cdot \left| \frac{z}{t} \right|^n \leq M \left( \frac{r}{t} \right)^n$$

Thus, letting  $\|\cdot\|$  be the uniform norm for  $C(\Delta)$ , we have  $\|c_n z^n\| \leq M(r/t)^n$ . Since  $r/t < 1$ ,  $\sum (r/t)^n < \infty$ , so by comparison  $\sum c_n z^n$  converges absolutely and uniformly in  $C(\Delta)$ .

Further, by definition,  $R$  is given by (i); the more esoteric formulation (ii) we shall leave as Problem 14.

### Examples

12. If  $\sum c_n z^n$  is a given power series with radius of convergence  $R$ , the question may arise: what happens on the circle  $|z| = R$ ? The answer is that practically anything can happen.

(a) If the sequence  $\{c_n\}$  is summable, that is,  $\sum |c_n| < \infty$ , then by comparison  $\sum c_n z^n$  converges uniformly in  $\{|z| \leq 1\}$ . Thus the series  $\sum (z^n/n^2)$  has radius of convergence 1 and converges uniformly in  $\{|z| \leq 1\}$ .

(b)  $\sum (z^n/n)$  also has radius of convergence 1, but  $\sum (1^n/n)$  does not converge, whereas  $\sum [(-1)^n/n]$  does converge.

(c)  $\sum z^n$  has radius of convergence 1, but  $\sum z^n$  does not converge for any  $z$  with  $|z| = 1$  at all ( $\lim z^n \neq 0$  if  $|z| = 1$ !).

Since no general assertion on the circle of convergence is possible, we needn't be concerned with the behavior of the series there (except in particular cases).

13. The geometric series  $\sum_{n=0}^{\infty} z^n$  is a power series with radius of convergence 1. This series converges to  $(1 - z)^{-1}$  uniformly and absolutely on any disk  $\{z \in C: |z| \leq r\}$  with  $r < 1$ . Thus

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \quad \text{for } |z| < 1$$

Now let  $a \in C$ ,  $a \neq 0$ . Then

$$\frac{1}{a - z} = \frac{1}{a} \cdot \frac{1}{[1 - (z/a)]} = \frac{1}{a} \sum_{n=0}^{\infty} \left( \frac{z}{a} \right)^n = \sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}}$$

This convergence is assured in the disk  $\{z \in C: |z| < |a|\}$ .

14. The series  $\sum_{n=0}^{\infty} (z^n/n!)$  has infinite radius of convergence.

Thus the sum is a continuous function on the whole plane, denoted  $e^z$  since this sum does converge to the exponential function for real values of  $z$ . We have seen that, for real  $x$ ,  $e^{ix} = \cos x + i \sin x$ . We can use this (or Taylor's theorem) to obtain series for the sine and cosine:

$$\begin{aligned} \cos x + i \sin x &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{x^{4k}}{(4k)!} + i \frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+2}}{(4k+2)!} - i \frac{x^{4k+3}}{(4k+3)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \end{aligned}$$

These series also converge on the entire plane. We can use them to define the complex cosine and sine:

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \quad \sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \quad (5.17)$$

We also have the equation

$$e^{iz} = \cos z + i \sin z \quad (5.18)$$

for *all* complex numbers  $z$  (for the series will sum again that way).

15. Replacing  $z$  by  $-iz$  and  $iz$ , alternately, we obtain these other interesting equations:

$$e^z = \cos(-iz) + i \sin(-iz) = \cos(iz) - i \sin(iz)$$

$$e^{-z} = \cos(iz) + i \sin(iz)$$

Thus

$$\frac{e^z + e^{-z}}{2} = \cos(iz) \quad (5.19)$$

$$\frac{e^z - e^{-z}}{2} = -i \sin(iz) \quad (5.20)$$

For real values of  $z$ , the left-hand sides of Equations (5.19), (5.20) are



the **hyperbolic cosine** and **hyperbolic sine**, respectively. We can use these expressions to define the **complex cosh** and **sinh**:

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

Because of (5.19) and (5.20), the complex trigonometric functions are, on the imaginary axis, the hyperbolic functions:

$$\cosh z = \cos(iz) \quad \sinh z = -i \sin(iz) \quad (5.21)$$

We should also note that the trigonometric identities imply the hyperbolic ones. Since  $\cos^2(iz) + \sin^2(iz) = 1$ , it follows from (5.21) that

$$\cosh^2 z - \sinh^2 z = 1 \quad (5.22)$$

(see Exercise 10).

16.  $\sum_{n=1}^{\infty} (z^n/n!)$  has radius of convergence 1. So does the series  $\sum_{n=1}^{\infty} n^k (z^n/n!)$ , for any integer  $k$ . We shall see later that the sums of all these series can be given by closed expressions (such as  $\sum z^n = (1-z)^{-1}$ ).

17. A polynomial function in  $C$  is given by a power series. In fact, writing the polynomial  $p(z) = \sum_{n=0}^N a_n z^n$  is the same as giving its power series expansion. What is more interesting is that any point in  $C$  can be chosen as the center of a power series expansion for  $p$ . Let  $z_0 \in C$  and write

$$p(z) = \sum_{n=0}^N a_n (z - z_0 + z_0)^n$$

Using the binomial theorem this becomes

$$p(z) = \sum_{n=0}^N a_n \sum_{i=0}^n \binom{n}{i} (z - z_0)^i z_0^{n-i} \quad (5.23)$$

All sums being finite, we may arrange terms at will. Thus we can rewrite (5.23) as a sum of powers of  $z - z_0$ :

$$p(z) = \sum_{n=0}^N \sum_{m=n}^N a_n \binom{m}{n} z_0^{m-n} (z - z_0)^n$$

which is the desired expansion.

More generally, any series of the form  $\sum_{n=0}^{\infty} c_n(z - z_0)^n$  will be called a power series expansion **centered at**  $z_0$ . Can we expand  $e^z$  in a power series centered at a point other than the origin? The answer is *yes* (cf., Problem 15), and the proof is like the one above for polynomials, but the question of convergence intervenes after the analog of Equation (5.23) above. It is a general fact that for *any* function given by a power series, we may move the center of the expansion to any other point in the disk of convergence. The truly courageous student should try to prove this now; it can be done. We will give a proof later which is simple and avoids convergence problems but requires more sophisticated information about functions defined by power series expansions.

### Addition and Multiplication of Power Series

Suppose  $f, g$  are complex-valued functions defined by power series expansions centered at a point  $z_0$ . Then we can find series expansions for the functions  $f + g$  and  $fg$  also. Addition is easy: if, say

$$f(z) = \sum a_n(z - z_0)^n \quad g(z) = \sum b_n(z - z_0)^n$$

then

$$(f + g)(z) = \sum (a_n + b_n)(z - z_0)^n$$

But to find the series expansion for the product requires a little more care. Suppose that  $z_0 = 0$  (this involves no loss of generality). To say that  $f(z) = \sum a_n z^n$  is to say that in a certain disk  $\Delta$ ,  $f$  is the limit of the polynomials  $\sum_{n=0}^N a_n z^n$ . Similarly,  $g$  is the limit of the polynomials  $\sum_{n=0}^N b_n z^n$ . Thus,  $fg$  is the limit of the sequence of polynomials  $(\sum_{n=0}^N a_n z^n)(\sum_{n=0}^N b_n z^n)$ . Now, we can multiply polynomials easily,

$$\left( \sum_{n=0}^N a_n z^n \right) \left( \sum_{n=0}^N b_n z^n \right) = \sum_{n=0}^N \sum_{m=0}^N a_n b_m z^{n+m}$$

If we collect terms in this expression to form a series of powers of  $z$  we do not get a very aesthetic expression, but if we take some terms from the next few polynomials in the sequence we obtain a reasonable expression.

$$\sum_{k=0}^{2N} \left( \sum_{n+m=k} a_n b_m \right) z^k \quad (5.24)$$

We could hope that  $fg$  is the limit of this sequence of polynomials. This is a reasonable hope; for even though we have modified the original sequence

of polynomials we have neither added nor deleted from the series represented by that sequence. In fact, by making careful use of this fact, we can verify that  $fg$  is the limit of (5.21).

**Proposition 3.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ,  $g(z) = \sum_{n=0}^{\infty} b_n z^n$  and suppose  $r$  is less than the radii of convergence of both series. Then*

- (i)  $(f + g)(z) = \sum_{n=0}^{\infty} (a_n + b_n)z^n$  uniformly and absolutely in  $\Delta = \{z \in C: |z| \leq r\}$ ,
- (ii)  $(fg)(z) = \sum_{k=0}^{\infty} (\sum_{n+m=k} a_n b_m)z^k$  uniformly and absolutely in  $\Delta$ .

*Proof.* Let  $p_n(z) = \sum_{k=0}^n a_k z^k$ ,  $q_n(z) = \sum_{k=0}^n b_k z^k$ . By hypothesis  $p_n \rightarrow f$ ,  $q_n \rightarrow g$  uniformly in  $\Delta$ . Thus  $p_n + q_n \rightarrow f + g$ ,  $p_n q_n \rightarrow fg$  uniformly in  $\Delta$  (Problem 2.55). Since

$$p_n(z) + q_n(z) = \sum_{k=0}^n (a_k + b_k)z^k$$

(i) is proven.

(ii) Let

$$r_n(z) = \sum_{k=0}^n \left( \sum_{i+j=k} a_i b_j \right) z^k$$

we want to show that  $r_n \rightarrow fg$  uniformly in  $\Delta$ . We know that  $p_n q_n \rightarrow fg$  so it would seem worth our while to compute  $p_n q_n - r_n$ . But that is easy,

$$p_n q_n - r_n = \sum_{k=0}^{2n} \left( \sum_{\substack{i+j=k \\ i > n \\ j > n}} a_i b_j \right) z^k$$

Now, each term on the right is of the form  $a_i b_j z^{i+j}$  with  $i > n$  or  $j > n$ . Thus, computing norms on  $\Delta = \{z \in C: |z| \leq r\}$ ,

$$\begin{aligned} \|p_n q_n - r_n\| &\leq \left( \sum_{i=n+1}^{2n} \|a_i z^i\| \right) \left( \sum_{j=0}^{2n} \|b_j z^j\| \right) \\ &\quad + \left( \sum_{i=0}^{2n} \|a_i z^i\| \right) \left( \sum_{j=n+1}^{2n} \|b_j z^j\| \right) \\ &\leq \left( \sum_{i=n+1}^{\infty} |a_i| r^i \right) \left( \sum_{j=0}^{\infty} |b_j| r^j \right) + \left( \sum_{i=0}^{\infty} |a_i| r^i \right) \left( \sum_{j=n+1}^{\infty} |b_j| r^j \right) \end{aligned}$$

Now, we know that  $\sum_{i=0}^{\infty} |a_i| r^i$ ,  $\sum_{j=0}^{\infty} |b_j| r^j$  are finite. Let  $M$  be a number larger than both. Given  $\varepsilon > 0$ , there are

- (1)  $N_1 > 0$  such that  $\sum_{i=n+1}^{\infty} |a_i| r^i < \varepsilon$  if  $n \geq N_1$ ,
- (2)  $N_2 > 0$  such that  $\sum_{j=n+1}^{\infty} |b_j| r^j < \varepsilon$  if  $n \geq N_2$ ,
- (3)  $N_3 > 0$  such that  $\|p_n q_n - fg\| < \varepsilon$  if  $n \geq N_3$ .

These assertions follow from the known convergence of each case. Thus, if  $n \geq \max(N_1, N_2, N_3)$ ,

$$\begin{aligned} \|r_n - fg\| &\leq \|p_n q_n - fg\| + \|p_n q_n - r_n\| \\ &< \varepsilon + \left( \sum_{i=n+1}^{\infty} |a_i| r^i \right) \left( \sum_{i=0}^{\infty} |b_j| r^j \right) + \left( \sum_{i=0}^{\infty} |a_i| r^i \right) \left( \sum_{j=n+1}^{\infty} |b_j| r^j \right) \\ &< \varepsilon + \varepsilon \cdot M + M \cdot \varepsilon = (2M + 1)\varepsilon \end{aligned}$$

the proposition is concluded.

### ● EXERCISES

9. Verify, in the way suggested in the text that  $\cos^2 z + \sin^2 z = 1$  is true for all complex numbers  $z$ , and thus  $\cosh^2 z - \sinh^2 z = 1$  is always true.

10. Find a power series expansion for these functions:

(a)  $\exp(z^2)$                       (e)  $e^{-x} \int_0^x \frac{e^t}{1-t} dt$

(b)  $e^z \sin z$                       (f)  $\cosh z$

(c)  $\frac{\cos z}{1-z}$                       (g)  $\sinh z$

(d)  $\int_0^x \exp(t^2) dt$

11. Verify by multiplying the power series that  $e^{z+w} = e^z e^w$ .

12. From the addition formula for the exponential (Exercise 11), deduce the addition formulas for  $\cos$ ,  $\sin$ ,  $\sinh$ ,  $\cosh$ .

### ● PROBLEMS

14. If  $\{c_n\}$  is any bounded sequence, then the maximum number, every neighborhood of which has infinitely many members, is denoted  $\limsup c_n$ . Show that the radius of convergence of the series  $\sum c_n z^n$  is  $R = (\limsup(|c_n|)^{1/n})^{-1}$ .

15. Expand  $e^z$  in a power series about any point  $z_0$ .

16. Assuming that  $(1+x^2)^{-1/2}$  can be represented by a power series centered at 0, find it. Find the power series for  $\arccos x$ .

17. Assuming that  $\tan x$  can be represented by a power series centered at 0 and using the equation  $\tan x \cos x = \sin x$ , find the power series expansion of  $\tan x$ .

## 5.6 Complex Differentiation

An easy property of the exponential function is

$$\lim_{z \rightarrow 0} \frac{e^z - 1}{z} = 1 \quad (5.25)$$

For

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

so

$$\frac{e^z - 1}{z} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = 1 + z \left( \sum_{n=2}^{\infty} \frac{z^{n-2}}{n!} \right)$$

Now the term in parenthesis is a convergent power series, so is continuous at 0. Thus, writing the parenthesis as  $g(z)$ :

$$\lim_{z \rightarrow 0} \frac{e^z - 1}{z} = 1 + \lim_{z \rightarrow 0} zg(z) = 1$$

From (5.25) and the properties of the exponential it follows that

$$\lim_{z \rightarrow 0} \frac{\exp(z_0 + z) - \exp(z_0)}{z} = \exp(z_0) \lim_{z \rightarrow 0} \frac{e^z - 1}{z} = \exp(z_0)$$

The student of calculus will recognize the limit on the left as a difference quotient and the entire equation as a replica of the behavior of the real exponential function. It might be a good idea to consider more generally such a process of differentiation on the complex plane. This turns out to be a

very significant idea, because there are many beautiful and useful ways to represent functions which are so differentiable.

**Definition 3.** Let  $f$  be a complex-valued function defined in a neighborhood of  $z_0$  in  $C$ .  $f$  is **differentiable at  $z_0$**  if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case we write the limit as  $f'(z_0)$ . If  $f$  is defined in an open set  $U$  and differentiable at every point of  $U$ , we say that  $f$  is **differentiable on  $U$** .

The usual algebraic facts on differentiation hold true in the complex domain.

**Proposition 4.**

(i) Suppose  $f, g$  are differentiable at  $z_0$ . Then so are  $f + g$  and  $fg$  with the derivatives given by

$$\begin{aligned}(f + g)'(z_0) &= f'(z_0) + g'(z_0) \\ (fg)'(z_0) &= f'(z_0)g(z_0) + f(z_0)g'(z_0)\end{aligned}$$

(ii) Suppose  $f$  is differentiable at  $z_0$  and  $f(z_0) \neq 0$ . Then  $1/f$  is differentiable at  $z_0$  and  $(1/f)'(z_0) = -f'(z_0)/f(z_0)^2$ .

(iii) Suppose  $f$  is differentiable at  $z_0$  and  $g$  is differentiable at  $f(z_0)$ . Then  $g \circ f$  is differentiable at  $z_0$  and  $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$ .

*Proof.* These propositions are so much like the corresponding propositions in calculus that their proofs will be left to the reader.

**Examples**

18. The function  $z$  is clearly differentiable, and  $z'(z_0) = 1$  for all  $z_0$ . A constant function is differentiable with derivative zero. Since any polynomial is obtained from  $z$  and constant functions by a succession of operations as described in Proposition 4(i), all polynomials are differentiable.

19. The function  $\bar{z}$  is nowhere differentiable. For the difference quotient  $(\bar{z} - \bar{z}_0)/(z - z_0)$ , for  $z \neq z_0$ , is a point on the unit circle and as  $z$  ranges through a neighborhood of  $z_0$  this difference quotient takes on all values on the unit circle, so it could hardly converge.

*Sum of a Power Series is Infinitely Differentiable*

Our introduction to this section was essentially a proof that  $e^z$  is everywhere differentiable. It is in fact true that the sum of a power series is differentiable in its disk of convergence. We now verify this basic fact.

**Theorem 5.7.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  have radius of convergence  $R$ . Then  $f$  is differentiable at every point in the disk of radius  $R$  and*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \quad (5.26)$$

*has the same radius of convergence as  $\sum_{n=0}^{\infty} a_n z^n$ .*

*Proof.*  $\limsup(n|a_n|)^{1/n} = \lim(n)^{1/n} \limsup(|a_n|)^{1/n} = \limsup(|a_n|)^{1/n}$ , so the series  $\sum n a_n z^{n-1}$  has the same radius of convergence as the given series. We must show that it represents the derivative of  $f$ . Fix a  $z_0$ ,  $|z_0| < R$  and choose  $r > |z_0|$ . The series  $\sum n |a_n| r^{n-1}$  converges absolutely, so given  $\varepsilon > 0$  there is an  $N$  such that  $\sum_{n>N} n |a_n| r^{n-1} < \varepsilon$ . Now consider the difference quotient defining  $f'(z_0)$ :

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} &= \sum_{n=0}^{\infty} a_n \left( \frac{z^n - z_0^n}{z - z_0} \right) \\ &= \sum_{n=1}^{\infty} a_n \left( \sum_{k=1}^n z^{n-k} z_0^{k-1} \right) \end{aligned}$$

If  $|z| < r$  as well as  $|z_0| < r$ , then

$$\left| \sum_{n>N} a_n \left( \sum_{k=1}^n z^{n-k} z_0^{k-1} \right) \right| \leq \sum_{n>N} |a_n| \sum_{k=1}^n r^{n-k} r^{k-1} \leq \sum_{n>N} n |a_n| r^{n-1} < \varepsilon$$

Similarly,  $|\sum_{n>N} n a_n z_0^{n-1}| < \varepsilon$ . Thus,

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} n a_n z_0^{n-1} \right| \leq 2\varepsilon + \sum_{n=1}^N |a_n| \left| \sum_{k=1}^n z^{n-k} z_0^{k-1} - z_0^{n-1} \right|$$

Now, by continuity, as  $z \rightarrow z_0$  the last term tends to zero. Thus, there is a  $\delta > 0$  such that if  $|z - z_0| < \delta$ , the last term is less than  $\varepsilon$ . Thus, for  $|z| < r$  and  $|z - z_0| < \delta$  we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} n a_n z_0^{n-1} \right| < 3\varepsilon$$

which proves that the limit of the difference quotient exists and is given by (5.26).

In particular, since  $f'$  is given by a convergent power series, it also is differentiable, with derivative  $f''(z) = \sum n(n-1)a_n z^{n-2}$ , and so forth. We thus obtain these results, which form the complex version of Taylor's theorem for sums of convergent power series.

**Corollary 1.** *Let  $f(z) = \sum a_n z^n$  have radius of convergence  $R$ . Then  $f$  is infinitely differentiable in  $\{z: |z| < R\}$ . Furthermore, for every  $k$  the  $k$ th derivative,  $f^{(k)}$  is given by a convergent power series,*

$$f^{(k)}(z) = \sum_{n>k} n(n-1) \cdots (n-k+1) a_n z^{n-k}$$

**Corollary 2.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be convergent in a disk about 0. The coefficients  $\{a_n\}$  are uniquely determined by  $f$ :*

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Notice that the definition of complex derivative is a genuine generalization of the differentiation of functions of a real variable. Thus the same corollaries hold for functions of a real variable represented by power series:

**Corollary 3.** *If  $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$  in a neighborhood of  $x_0$ , then  $f$  is infinitely differentiable at  $x_0$  and*

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

$$f^{(k)}(x) = \sum_{n>k} n(n-1) \cdots (n-k) a_n (x - x_0)^{n-k}$$

These corollaries are easily derived from the theorem and their proofs are left to the student. Notice that the implication of Corollary 2 is that the coefficients of a power series representation of a function are uniquely and directly determined by the function. In particular, a function cannot be written as the sum of a power series in more than one way. This observation allows us to easily verify the identity

$$\cos^2 z + \sin^2 z = 1 \tag{5.27}$$

For the function  $\cos^2 z + \sin^2 z$  is a polynomial in functions which are sums of power series and thus is the sum of a power series. Its coefficients can



be computed according to Corollary 3 just by letting  $z$  take on real values. But the right-hand side of (5.27) is the Taylor expansion of  $\cos^2 z + \sin^2 z$ , for real  $z$ , thus it must be the Taylor expansion for all  $z$ . Hence (5.27) is always true.

### The Cauchy-Riemann Equation

It is of value to compare the notion of complex differentiation with that of differentiation of functions defined on  $R^2$ , since  $R^2 = C$ . Suppose that  $f$  is a complex-valued function defined in a neighborhood of  $z_0 = x_0 + iy_0$ . If  $f$  is differentiable as a function of two real variables, then the differential  $df(x_0, y_0)$  is defined and is a complex-valued linear function on  $R^2$ . If  $f$  is also complex differentiable, then

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad (5.28)$$

exists. Let  $z \rightarrow z_0$  along the horizontal line. Then (5.28) specializes to

$$f'(z_0) = \lim_{x \rightarrow x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \frac{\partial f}{\partial x}(x_0, y_0) \quad (5.29)$$

If we let  $z \rightarrow z_0$  along a vertical, we also have

$$f'(z_0) = \lim_{y \rightarrow y_0} \frac{f(x_0, y) - f(x_0, y_0)}{i(y - y_0)} = \frac{1}{i} \frac{\partial f}{\partial y}(x_0, y_0) \quad (5.30)$$

Thus the right-hand sides of (5.29) and (5.30) are the same. In conclusion, a complex differentiable function must satisfy (when considered as a function of two real variables) this relation

$$\left(\frac{\partial f}{\partial x}\right) = -i\left(\frac{\partial f}{\partial y}\right) \quad (5.31)$$

This is called the **Cauchy-Riemann equation**. More precisely, the Cauchy-Riemann equations are found by writing  $f = u + iv$  and splitting into real and imaginary parts. Let us record this important fact.

**Theorem 5.8.** *Let  $f$  be a complex differentiable function in a domain  $D$ . Split  $f$  into real and imaginary parts and consider  $f$  as a function of two real*

variables ( $z = x + iy$ ). Then these partial differential equations hold in  $D$ :

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} \quad (5.32)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (5.33)$$

*Proof.* Equation (5.32) was observed above. Equation (5.33) follows from (5.32) and the identities

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$$

Notice that when  $f$  is complex differentiable, its differential is given by

$$\begin{aligned} df(z_0, (r + is)) &= \frac{\partial f}{\partial x}(z_0)r + \frac{\partial f}{\partial y}(z_0)s \\ &= f'(z_0)r + if'(z_0)s \\ &= f'(z_0)(r + is) \end{aligned}$$

Thus the differential of a complex differentiable function is a **complex linear** complex-valued function.

We shall show, via the techniques of the next few chapters, that a complex differentiable function can be written as the sum of a convergent power series. Thus, just by virtue of the differential being complex linear, the function *has derivatives of all orders and is the sum of its power series*.

## ● PROBLEMS

18. Prove Proposition 4.
19. Prove Corollaries 1 and 2 of Theorem 5.7.
20. Show that if  $f$  is an infinitely differentiable function on an interval  $(-\varepsilon, \varepsilon)$ , and there is an  $M > 0$  such that

$$|f^{(n)}(x)| \leq M \quad \text{all } n \quad \text{all } x \quad -\varepsilon < x < \varepsilon$$

then  $f$  is the sum of a power series which converges in the unit disk.

21. Suppose  $f$  is a complex differentiable function in a domain in the plane. Show that:

- (a) if  $f$  is real-valued it is constant.
- (b) if  $|f|$  is constant, then  $f$  is constant.

22. Write the Cauchy-Riemann equations in polar coordinates. (*Hint:* Differentiate along the ray and circle through a point.)

23. Suppose  $f_1, \dots, f_k$  are given by convergent power series in a disk  $\Delta$ . If  $F$  is a polynomial in  $k$  variables such that

$$F(f_1(z), \dots, f_k(z)) = 0 \quad (5.34)$$

for real  $z$ , then (5.34) is true for all  $z$  in  $\Delta$ .

24. Compute the limits of these quotients as  $z \rightarrow 0$ :

$$(a) \frac{\arctan z}{z}$$

$$(d) \frac{\cos z - 1}{z^2}$$

$$(b) \frac{\sin hz}{\sin z}$$

$$(e) \frac{\cos z}{1 + z^2}$$

$$(c) \frac{\cos z - 1}{z}$$

$$(f) \frac{\sin z - \tan z}{z^n} \quad n = 0, 1, 2, 3, 4$$

25. Suppose  $f$  is a differentiable complex-valued function of two real variables in a domain  $D$ . Show that  $f$  is a complex differentiable if and only if the differential  $df(z_0)$  is complex linear for all  $z_0 \in D$ .

26. Suppose that  $f$  is twice differentiable in  $D$ , and is complex differentiable. Show also that  $f'$  is complex differentiable. Supposing that  $(f')' \equiv 0$ , show that  $f$  is a quadratic polynomial in  $z$ .

27. If  $f = u + iv$  is complex differentiable in  $D$  and twice differentiable, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

## 5.7 Differential Equations with Analytic Coefficients

A function which can be represented as the sum of a convergent power series at a point  $a \in C$  will be said to be **analytic** at  $a$ . We now return to the study of linear differential equations in order to answer some of the questions posed in Section 5.5. We can use the information in Section 5.6 to do this and to provide the sought-for estimates. In particular, we shall verify the following fact.

**Proposition 5.** Suppose  $h, g_0, \dots, g_{k-1}, g_k$  are analytic at 0. Then the solution of the differential equation

$$y^{(k)} + \sum_{i=0}^{k-1} g_i y^{(i)} + h(x) = 0 \quad y(0) = a_0, \dots, y^{(k-1)}(0) = a_{k-1}$$

is also analytic at 0; that is, in some disk centered at 0, it is the sum of a convergent power series whose coefficients can be recursively calculated from the differential equation.

We already know, from Section 5.5, how to compute the Taylor coefficients of the solution; our business here is to show that the resulting series does in fact converge. This, of course, involves producing the kind of estimate required by Theorem 5.7.

Suppose then that  $h, g_0, \dots, g_{k-1}$  are analytic in the interval  $|x| \leq R$ . Then

$$h(x) = \sum_{n=0}^{\infty} a_n x^n \quad g_i(x) = \sum_{n=0}^{\infty} a_n^i x^n$$

and for some positive number  $M$ ,  $|a_n| \leq MR^{-n}$ ,  $|a_n^i| \leq MR^{-n}$  for all  $i$  and  $n$ . We shall obtain the desired estimate in terms of  $M$ ,  $R$  and the initial conditions. For simplicity, we shall do the homogeneous case only ( $h = 0$ ), leaving the general case for the reader (Problem 27). If

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

is the desired solution, we have

$$c_0 = a_0, \dots, c_{k-1} = \frac{a_{k-1}}{(k-1)!}$$

and the rest of the coefficients are found from these equations:

$$\sum_{n=k}^{\infty} n(n-1) \cdots (n-k) c_n x^{n-k} + \sum_{i=0}^{k-1} \left( \sum_{n=0}^{\infty} a_n^i x^n \right) \left( \sum_{n=1}^{\infty} n(n-1) \cdots (n-i) c_n x^{n-i} \right) = 0 \quad (5.35)$$

Surely, the reader now has a pain in his stomach similar to that of the author as he wrote this equation. Patience, dear reader—the fun has just begun! Equating coefficients of  $x^n$  to zero, we obtain this recursive system of equations for the

coefficients:

$$m(m-1) \cdots (m-k)c_m + \sum_{i=0}^{k-1} \sum_{\alpha=0}^{m-k} a_\alpha^i (m+i-(k+\alpha)) \cdots (m-(k+\alpha))c_{m+i-(k+\alpha)} = 0$$

or

$$c_m = \frac{-1}{m \cdots (m-k)} \sum_{i=0}^{k-1} \sum_{\alpha=0}^{m-k} (m+i-(k+\alpha)) \cdots (m-(k+\alpha)) a_\alpha^i c_{m+i-(k+\alpha)} \quad (5.36)$$

By the restraints on  $i$  and  $\alpha$  we have  $m+i-(k+\alpha) \leq m+k-1-k = m-1$ , so the highest subscript of  $c$  on the right is  $m-1$ . Thus, given  $c_0, \dots, c_{k-1}$  we can solve Equations (5.36) successively. We now try to find an estimate. For this purpose assume  $M \geq 1$ , and let

$$C = \max \left( K, R(R-1)^{-1}(R^k-1), \frac{R}{M}, \left( \frac{|a_j|}{j!} \right)^{1/j}, 0 \leq j \leq k-1 \right)$$

The last condition on  $C$  is written so as to assure that

$$|c_j| \leq \left( \frac{CM}{R} \right)^j \quad \text{for } j=0, \dots, k-1 \quad (5.37)$$

We now prove this inequality for all  $m$ , by induction. Thus we use (5.36), assuming (5.37) for all  $n < m$ :

$$\begin{aligned} |c_m| &\leq \frac{1}{m} \sum_{i=0}^{k-1} \sum_{\alpha=0}^{m-k} |a_\alpha^i| |c_{m+i-(k+\alpha)}| \\ &\leq \frac{1}{m} \sum_{i=0}^{k-1} \sum_{\alpha=0}^{m-k} \frac{M}{R^\alpha} \left( \frac{CM}{R} \right)^{m+i-(k+\alpha)} \\ &\leq \frac{1}{m} \frac{M^m C^{m-1}}{R^{m-k}} \sum_{i=0}^{k-1} \frac{(m-k+1)}{R^i} \end{aligned}$$

Now  $\sum_{i=0}^{k-1} R^{-i} = (R^{-k}-1)(R^{-1}-1)^{-1} = R(R-1)^{-1}(R^k-1)R^{-k}$ . Thus

$$|c_m| \leq \frac{m-k+1}{m} \frac{M^m}{R^m} C^{m-1} \frac{R(R^k-1)}{R-1} \leq \left( \frac{CM}{R} \right)^m$$

by definition of  $C$ . By this estimate we see that  $f(x) = \sum_{k=0}^\infty c_n x^n$  converges in the interval  $\{x: |x| < R(CM)^{-1}\}$ , and in that interval is the solution to our problem.

We might perhaps have made a better estimate by more clever substitutions; but our above estimates were sufficient for the results desired. In any particular case we could usually be clever and obtain even better estimates.

### Examples

$$20. \quad y' + e^x y' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

We will find a polynomial which approximates the solution to within  $10^{-5}$  on the interval  $[0.01, 0.01]$ . Let  $\sum c_n x^n$  be the supposed solution. By substituting the series we obtain

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left( \sum_{n=1}^{\infty} n c_n x^{n-1} \right) + \sum_{n=0}^{\infty} c_n x^{n+1} = 0 \quad (5.38)$$

The initial conditions give  $c_0 = 0$ ,  $c_1 = 1$ . Equation (5.38) becomes

$$c_m = \frac{-1}{m(m-1)} \left( \sum_{i=0}^{m-2} \frac{1}{i!} (m-1-i)c_{m-1-i} + c_{m-3} \right)$$

Thus

$$c_2 = -\frac{1}{2}c_1 = -\frac{1}{2}$$

$$c_3 = -\frac{1}{6}(2c_2 + c_1 + c_0) = 0$$

$$c_4 = -\frac{1}{12}(3c_3 + 2c_2 + \frac{1}{2}c_1 + c_1) = \frac{1}{24}$$

$$c_5 = -\frac{1}{20}(4c_4 + 3c_3 + c_2 + \frac{1}{6}c_1 + c_2) = \frac{1}{120}$$

and so forth. The question is not really what the coefficients are (that is to be left to a machine)—but how many coefficients need to be computed. The coefficients appear to be bounded (we could in fact show that they must be, cf. Problem 29). Let's try to prove that  $|c_n| \leq K$  for all  $n$  by induction. We have

$$\begin{aligned} |c_m| &\leq \frac{1}{m(m-1)} \sum_{i=0}^{m-2} \frac{1}{i!} (m-1-i)|c_{m-1-i}| + |c_{m-3}| \\ &\leq \frac{K}{m} \left( \sum_{i=0}^{m-2} \frac{1}{i!} + 1 \right) \leq \frac{K(e+1)}{m} \leq K \end{aligned}$$

so long as  $m \geq 4$ . Thus we may take for  $K$  a bound for the first four terms, that is,  $k = 1/2$ . Then the difference between the solution

and the  $k$ th partial sum of its Taylor expansion is dominated by

$$\sum_{n \geq k} |c_n| |x|^n \leq \frac{1}{2} \sum_{n \geq k} |x|^n = \frac{1}{2} |x|^k \frac{1}{1 - |x|}$$

for  $|x| < 1$ . The interval we are concerned with is  $|x| < 10^{-1}$  so our bound on the error is

$$\frac{1}{2} \cdot \frac{1}{10^k} \cdot \frac{10}{9} = \frac{1}{18} 10^{-k+1}$$

This is less than  $10^{-5}$  if  $k = 6$ , thus (computing also  $c_6$ ) the solution differs from

$$x - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{360}x^6$$

by at most  $10^{-5}$  for all values of  $x$  in  $[-0.01, 0.01]$ .

21. Suppose we needed that good an estimate in the interval  $[-1, 1]$ . It is easy to see that just knowing that the coefficients are bounded is not good enough. We have to know that  $|c_n| \leq Kr^n$  for some  $r < 1$  and some  $K$ . Let's try  $r = \frac{1}{2}$ . That is, we attempt to verify by induction that  $|c_n| \leq 2^{-n}$  for all  $n$ . Now, using the equation defining  $\{c_n\}$ ,

$$\begin{aligned} |c_m| &\leq \frac{1}{m(m-1)} \left( \sum_{i=0}^{m-2} \frac{(m-1-i)}{i!} \cdot \frac{K}{2^{m-1-i}} + \frac{K}{2^{m-3}} \right) \\ &\leq \frac{K}{m} \cdot \frac{1}{2^{m-1}} \left( \sum_{i=0}^{m-2} \frac{2^i}{i!} + 4 \right) \\ &\leq \frac{e^2 + 4}{m} \cdot \frac{K}{2^{m-1}} \end{aligned}$$

This is less than  $2^{-m}K$  as soon as  $m \geq 2(e^2 + 4)$ , or  $m \geq 26$ . Thus the induction step proceeds as soon as  $m \geq 26$ ; we need only choose  $K$  so that the inequality holds for all  $m < 26$  ( $K = 2$  will do). Thus  $|c_n| < 1/2^{n-1}$  for all  $n$ . The desired solution differs in the interval  $[-1, 1]$  from its  $k$ th-order Taylor polynomials by

$$\left| \sum_{n > k} c_n(1)^n \right| \leq \sum_{n > k} |c_n| \leq \frac{1}{2^{k-1}} \sum_{n=0}^{\infty} \frac{1}{2^n} \leq \frac{1}{2^{k-2}}$$

This is less than  $10^{-5}$  if  $k$  is 19. Thus we need to compute 19 terms of the Taylor expansion to find the desired approximation.

22. Compute the solution of

$$y'' + \left(\frac{1}{1-x}\right)y' + e^x y = 0 \quad y(0) = 1 \quad y'(0) = 0$$

to an accuracy of  $10^{-3}$  in the interval  $[-\frac{1}{2}, \frac{1}{2}]$ .

Let  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  be the solution. We have these equations for the solution:

$$\begin{aligned} c_0 &= 1, \quad c_1 = 0 \\ &\vdots \\ c_n &= \frac{-1}{n(n-1)} \left( \sum_{i=0}^{n-2} (n-1-i)c_{n-1-i} + \sum_{i=0}^{n-2} \frac{1}{i!} c_{n-2-i} \right) \end{aligned} \quad (5.39)$$

We will show by inductions that the  $\{c_n\}$  are bounded. Suppose that  $|c_n| \leq K$  for all  $n < m$ . Then

$$\begin{aligned} |c_m| &\leq \frac{1}{m(m-1)} \left( \sum_{i=0}^{m-2} (m-1-i)K + \sum_{i=0}^{m-2} \frac{1}{i!} K \right) \\ &\leq \frac{K}{m(m-1)} \left( \sum_{j=1}^{m-1} j + e \right) = \frac{K}{m(m-1)} \left[ \frac{m(m-1)}{2} + e \right] \\ &= K \left[ \frac{1}{2} + \frac{e}{m(m-1)} \right] \leq K \end{aligned}$$

as soon as  $m \geq 3$ . Thus we can take  $K$  as a bound for the first four terms. We have from (5.39)  $c_2 = -\frac{1}{2}$ ,  $c_3 = 0$ , so we may take  $K = 1$ . Then the estimate of the remainder after  $k$  terms in the interval  $[-\frac{1}{2}, \frac{1}{2}]$  is

$$\sum_{n>k} |c_n| |x|^n \leq \sum_{n>k} \frac{1}{2^n} = \frac{1}{2^k}$$

This is less than  $10^{-3}$  when  $k = 10$ , so we need 11 coefficients. We compute

$$c_4 = \frac{1}{12} \quad c_5 = \frac{1}{20} \quad c_6 = \frac{1}{720} \quad \text{etc.}$$



Up to six terms (giving at most an error of  $1/64$ ), our solution is

$$1 - \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^5}{20} + \frac{13x^6}{720} + \cdots$$

### ● EXERCISES

13. Find a power series expansion for the general solution in a neighborhood of 0 for this equation,

$$(1 - x^2)y'' - 2xy' + k(k+1)y = 0$$

14. Find a power series expansion for the general solution of

$$y'' - 2xy' + 2ky = 0$$

15. Find the power series for the function  $y = f(x)$  such that

(a)  $y'' + e^x y = x^2$ ,  $y(0) = 1$ ,  $y'(0) = 0$

(b)  $(y')^2 = y$ ,  $y(0) = 0$

(c)  $(y')^2 = yy''$ ,  $y(0) = 0$ ,  $y'(0) = 1$

(d)  $y' + 2xy^2 = 0$

16. How many terms of the power series for the solution  $y$  do we need:

(a) for an accuracy of  $10^{-3}$  in the interval  $(-\frac{1}{2}, \frac{1}{2})$  in Exercise 3(a)?

(b) for an accuracy of  $10^{-5}$  in the interval  $(-10, 10)$  in Exercise 3(a)?

(c) for an accuracy of  $10^{-3}$  in the interval  $(-0.1, 0.1)$  in Exercise 3(b)?

17. For what  $k$  are the solution of Equations (5.1), (5.2) polynomials?

### ● PROBLEMS

28. Generalizing the argument in the text prove this theorem:

**Theorem.** *If  $h, g_0, \dots, g_{k-1}$  are analytic in an interval  $(-R, R)$  about the origin, then any solution of the differential equation*

$$y^{(k)} + \sum_{r=0}^{k-1} g_r y^{(r)} = h$$

*can be expressed as the sum of a convergent power series in a neighborhood of the origin.*

29. Suppose that  $g, h$  are convergent power series in some disk  $\{|z| \leq R\}$  with  $R > 1$ . Show that the solution of the linear differential equation

$$y'' + gy' + hy = 0 \tag{5.40}$$

is the sum of a convergent power series with bounded coefficients.

30. If the power series  $g(x) = \sum a_n x^n$ ,  $h(x) = \sum b_n x^n$  both have infinite radius of convergence, then so does the series expansion of the solution of (5.40).

## 5.8 Infinitely Flat Functions

Not all functions are susceptible to the kind of Taylor series analysis which we have been doing. A first requirement is that the function have derivatives of all order; even that however is insufficient. Another glance at Theorem 5.6 will remind the reader that there is a behavior requirement on these successive derivatives in order that the given function be the sum of its Taylor expansion. We shall show by example that there are infinitely differentiable functions which are not sums of power series. First, we shall make the notion of analyticity precise.

**Definition 4.** Let  $f$  be a complex-valued function defined in an open set  $U$ . Let  $a \in U$ .  $f$  is **analytic at  $a$**  if there is a ball  $\{z: |z - a| < r\}$  centered at  $a$  such that  $f$  is the sum of a convergent power series in this ball.  $f$  is **analytic in  $U$**  if  $f$  is analytic at every point of  $U$ .

We have deliberately stated this definition without reference to the domain of definition of the function; it applies equally well to functions of a real or complex variable. The only functions which we know to be analytic are the polynomials and  $e^z$ . For example, if  $f$  is the sum of a convergent power series at the origin, we do not yet know that we can expand  $f$  in a series of powers of  $(z - a)$  with  $a$  any other point in the disk of convergence. We shall see in the next chapter that this is the case. We have already seen that an analytic function has derivatives of all orders (is  $C^\infty$ ) and now we will produce a  $C^\infty$  function which is not analytic. The clue to this function is given by the following fact, which follows from l'Hospital's rule.

**Proposition 6.**  $\lim_{t \rightarrow \infty} P(t)e^{-t} = 0$ , for any polynomial  $P$ .

*Proof.* Problem 31.

The function we have in mind (Figure 5.1) is defined by

$$\sigma(x) = \begin{cases} \exp\left(-\frac{1}{x}\right) & x > 0 \\ 0 & x \leq 0 \end{cases} \quad (5.41)$$

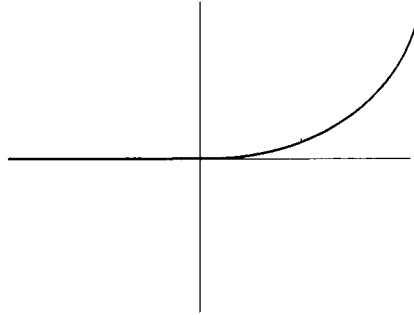


Figure 5.1

$\sigma$  is certainly infinitely differentiable at any point  $x_0 \neq 0$ , so we need only consider its behavior at 0. Now,

$$\sigma^{(n)}(x) = 0 \quad x < 0 \quad \text{all } n$$

Thus all derivatives of  $\sigma$  from the left exist at 0. We have to show that all derivatives from the right exist and are zero. More precisely we must prove that for all  $n$ ,

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{\sigma^{(n)}(x) - \sigma^{(n)}(0)}{x} = 0 \tag{5.42}$$

We do this by induction. The case  $n = 0$  is easy:

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{\sigma(x)}{x} = \lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{1}{x} \exp\left(-\frac{1}{x}\right) = \lim_{t \rightarrow \infty} te^{-t} = 0$$

To do the general case we must have some idea what  $\sigma^{(n)}(x)$  looks like for  $x > 0$ . Now

$$\begin{aligned} \sigma'(x) &= -\frac{1}{x^2} \exp\left(-\frac{1}{x}\right) & \sigma''(x) &= \left(\frac{2}{x^3} + \frac{1}{x^4}\right) \exp\left(-\frac{1}{x}\right) \\ \sigma^{(3)}(x) &= -\left(\frac{6}{x^4} + \frac{6}{x^5} + \frac{1}{x^6}\right) \exp\left(-\frac{1}{x}\right) \end{aligned}$$

A pattern seems to be developing.

For each  $n$  there is a polynomial  $P_n$  such that

$$\sigma^{(n)}(x) = P_n\left(\frac{1}{x}\right) \exp\left(-\frac{1}{x}\right) \quad \text{for } x > 0 \tag{5.43}$$

This can be verified by induction. Assuming (5.43), we compute

$$\begin{aligned} \sigma^{(n+1)}(x) &= P'_n\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2} \exp\left(-\frac{1}{x}\right) + P_n\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2} \exp\left(-\frac{1}{x}\right) \\ &= \frac{-1}{x^2} \left( P'_n\left(\frac{1}{x}\right) + P_n\left(\frac{1}{x}\right) \right) \exp\left(-\frac{1}{x}\right) \\ &= P_{n+1}\left(\frac{1}{x}\right) \exp\left(-\frac{1}{x}\right) \end{aligned}$$

where  $P_{n+1}(X) = -X^2(P'_n(X) + P_n(X))$ . Now that we have this, (5.42) follows immediately from Proposition 6:

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{\sigma^{(n)}(x) - \sigma^{(n)}(0)}{x} = \lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{1}{x} P_n\left(\frac{1}{x}\right) \exp\left(-\frac{1}{x}\right) = \lim_{t \rightarrow \infty} t P_n(t) e^{-t} = 0$$

Thus  $\sigma$  is also infinitely differentiable at 0. But it is certainly not analytic. Its Taylor expansion is  $\sum_{n=0}^{\infty} 0 \cdot x^n$  which converges to  $\sigma(x)$  only for  $x \leq 0$  and provides a poor means for approximating the value of  $\sigma(x)$  for  $x > 0$ .

However, the fact that infinitely differentiable functions exist with this property has its bright side. The following construction will prove to be useful.

**Lemma.** *Given  $a < b$ , there is a  $C^\infty$  function  $\tau_{ab}$  such that*

- (i)  $0 \leq \tau_{ab}(x) \leq 1$  for all  $x$ ,
- (ii)  $\tau_{ab}(x) > 0$  if  $a < x < b$ ,
- (iii)  $\tau_{ab}(x) = 0$  if  $x \geq b$  or  $x \leq a$ .

*Proof.* (See Figure 5.2.)  $\sigma(x(1-x))$  has the required properties of  $\tau_{01}$ . We then define

$$\tau_{ab}(x) = \tau_{01}\left(\frac{x-a}{b-a}\right) = \sigma\left(\left(\frac{x-a}{b-a}\right)\left(1 - \frac{x-a}{b-a}\right)\right)$$

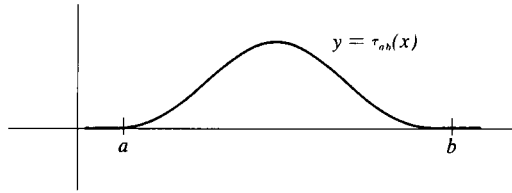


Figure 5.2

**Theorem 5.9.** Let  $[a, b]$  be a given interval and  $U_1, \dots, U_n$  a finite collection of open intervals covering  $[a, b]$ . There exist  $C^\infty$  functions  $\rho_i$  such that

- (i)  $0 \leq \rho_i(x) \leq 1$  for all  $x \in \mathbb{R}$ , all  $i$ ,
- (ii)  $\rho_i(x) = 0$  if  $x \notin U_i$ ,
- (iii)  $\sum \rho_i(x) = 1$ , for all  $x \in [a, b]$ .

*Proof.* Let  $U_i = (a_i, b_i)$ , and take  $\tau_i = \tau_{a_i b_i}$ . Then  $\tau(x) = \sum \tau_i(x) > 0$  if  $x \in (a, b)$ . Let

$$\rho_i(x) = \begin{cases} \frac{\tau_i(x)}{\tau(x)} & x \in (a_i, b_i) \\ 0 & x \notin (a_i, b_i) \end{cases}$$

The  $\rho_i$  then have the desired properties.

● PROBLEMS

- 31. Prove that for any polynomial  $P$ ,  $\lim_{t \rightarrow \infty} P(t)e^{-t} = 0$ .
- 32. Let  $\sigma$  be defined by (5.31). Define

$$\omega(x) = \frac{\int_0^x (t(1-t)) dt}{\int_0^1 (t(1-t)) dt}$$

Show that (a)  $\omega$  is  $C^\infty$ , (b)  $0 \leq \omega(x) \leq 1$ , for all  $x$ , (c)  $\omega(x) = 0$ , if  $x < 0$ , (d)  $\omega(x) = 1$ , if  $x \geq 1$ .

33. Using Theorem 5.9 it can be shown that any continuous function is the limit of  $C^\infty$  functions. Let  $f \in C([0, 1])$  and  $\varepsilon > 0$ . Find a  $C^\infty$  function  $g$  such that  $\|f - g\| < \varepsilon$ .

Here's how to do it. First pick an integer  $N > 0$  such that  $|f(x) - f(y)| < \varepsilon/2$  if  $|x - y| < 1/N$ . Now cover the interval  $[0, 1]$  by the intervals

$U_0, \dots, U_N$ , where

$$U_i = \left( \frac{i-1}{N}, \frac{i+1}{N} \right)$$

and let  $\rho_1, \dots, \rho_N$  be the corresponding functions of Theorem 5.9. Let

$$g(x) = \sum_{i=0}^N f\left(\frac{i}{N}\right) \rho_i(x)$$

For any  $x$ ,  $x$  is in only two of the intervals  $\{U_i\}$ , say  $U_r, U_{r+1}$ . Then

$$|f(x) - g(x)| = \rho_r(x) \left| f(x) - f\left(\frac{r}{N}\right) \right| + \rho_{r+1} \left| f(x) - f\left(\frac{r+1}{N}\right) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

## 5.9 Summary

Let  $\{f_k\}$  be a sequence of continuous functions. The *series* formed from the  $f_k$  is the sequence of sums  $\{\sum_{k=1}^n f_k\}$ . If this sequence converges, we say that the series converges and denote the limit by  $\sum_{k=1}^{\infty} f_k$ . The series *converges absolutely* if  $\sum_{k=1}^{\infty} \|f_k\| < \infty$ . The *Cauchy criterion* for series asserts that the series converges if and only if the sums  $\|\sum_{k=n+1}^m f_k\|$  can be made arbitrarily small by choosing  $m, n$  sufficiently large.

**COMPARISON TEST.** If there is a sequence  $\{p_k\}$  of positive numbers and an integer  $N > 0$  such that

- (i)  $\|f_k\| < p_k$  for  $k \geq N$
- (ii)  $\sum p_k < \infty$

then  $\sum f_k$  converges absolutely.

**INTEGRATION.** If  $\sum f_n$  converges, so does  $\sum \int_a^x f_n$  and

$$\int_a^x \left( \sum f_n \right) = \sum \left( \int_a^x f_n \right)$$

**FUNDAMENTAL THEOREM OF ALGEBRA.** If  $P$  is a polynomial:

$$P(z) = a_n z^n + \cdots + a_1 z + a_0 \tag{5.44}$$

with  $a_n \neq 0$ . Then  $P(z)$  has a complex root. If  $r_1, \dots, r_k$  are all the roots of  $P$ , then corresponding to each root there is a positive integer  $m_i$  (called the multiplicity) such that

$$\begin{aligned} \text{(i)} \quad & m_1 + \dots + m_k = n \\ \text{(ii)} \quad & P(z) = (z - r_1)^{m_1} \dots (z - r_k)^{m_k} \end{aligned} \quad (5.45)$$

To the polynomial  $P$  given by (5.44) we associate the *constant coefficient differential operator*  $L_P$ :

$$L_P(f) = a_n f^{(n)} + \dots + a_1 f' + a_0 f$$

$P$  is called the *characteristic polynomial* of  $L_P$ . These formulas are valid:

$$L_{P+Q} = L_P + L_Q \quad L_{PQ} = L_P L_Q$$

If (5.45) is the factorization of  $P$ , then the kernel of  $L_P$ , the collection of solutions of  $L_P(f) = 0$ , is spanned by the functions

$$e^{r_1 x}, \dots, x^{m_1-1} e^{r_1 x}$$

$$e^{r_2 x}, \dots, x^{m_2-1} e^{r_2 x}$$

$$e^{r_k x}, \dots, x^{m_k-1} e^{r_k x}$$

Let  $\{c_n\}$  be a sequence of complex numbers. There is a nonnegative number  $R$  (called the *radius of convergence of the power series*  $\sum c_n z^n$ ) with these properties:

- (a)  $\sum c_n z^n$  diverges for  $|z| \geq R$ .
- (b)  $\sum c_n z^n$  converges absolutely in  $\{|z| \leq r\}$  for  $r < R$ .
- (c)  $R = [\limsup(|c_n|)^{1/n}]^{-1}$ .

If  $f(z) = \sum a_n z^n$ ,  $g(z) = \sum b_n z^n$  in the disk  $\{|z| \leq r\}$ , then

$$f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$$

$$f(z)g(z) = \sum_{k=0}^{\infty} \left( \sum_{n+m=k} a_n b_m \right) z^k$$

in that same disk.

If  $f$  is a complex-valued function defined near  $z_0$  in  $C$ , we say that  $f$  is *complex differentiable at  $z_0$*  if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists. The sum of a convergent power series is complex differentiable at every  $z_0$  in its disk of convergence. Furthermore, the derivative is the sum of the derived series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

Thus the sum of a convergent power series is infinitely differentiable. A differentiable complex-valued function of two real variables is complex differentiable if and only if it satisfies the Cauchy–Riemann equations:

$$\frac{\partial f}{\partial x} = -i \left( \frac{\partial f}{\partial y} \right)$$

If  $h, g_0, \dots, g_k$  can be represented as sums of convergent power series in a disk centered at zero, then the same is true for all solutions of the differential equation

$$y^{(k)} + \sum_{i=0}^{k-1} g_i y^{(i)} + h = 0$$

Furthermore, once given the initial conditions

$$y(0) = a_0, \dots, y^{(k-1)}(0) = a_{k-1}$$

the coefficients of the power series can be recursively calculated using the differential equation.

Given any finite covering of the interval  $[a, b]$  by open intervals  $U_1, \dots, U_n$  we can find  $C^\infty$  functions  $\rho_1, \dots, \rho_n$  such that

- (a)  $0 \leq \rho_i \leq 1$
- (b)  $\rho_i = 0$ , outside  $U_i$
- (c)  $\sum \rho_i(x) = 1$  for all  $x \in [a, b]$

These functions are called a *partition of unity on  $[a, b]$  subordinate to the cover  $U_1, \dots, U_n$* .



● FURTHER READING

I. I. Hirshman, *Infinite Series*, Holt, Rinehart and Winston, New York, 1962.

H. Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Addison-Wesley, Reading, Mass., 1963.

This text develops the subject of complex analysis from the point of view of power series. It also contains a complete discussion of the theorem on existence of solutions of analytic differential equations.

Further material can be found in

T. A. Bak and J. Lichtenberg, *Mathematics for Scientists*, W. A. Benjamin, Inc., New York, 1966.

Kreider, Kuller, Ostberg, Perkins, *An Introduction to Linear Analysis*, Addison-Wesley, Reading, Mass., 1966.

W. Fulks, *Advanced Calculus*, John Wiley and Sons, New York, 1961.

M. R. Spiegel, *Applied Differential Equations*, Prentice-Hall, Englewood Cliffs, N. J., 1958.

● MISCELLANEOUS PROBLEMS

34. Find a sequence  $\{f_n\}$  of continuous nonnegative real-valued functions defined on the interval  $(0, 1)$  such that  $f(x) = \sum_{n=1}^{\infty} f_n(x)$  exists for all  $x \in [0, 1]$ , but  $f$  is not continuous.

35. For  $|z| < 1$ , define

$$\ln z = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

Show that for all such  $z$ ,  $z = 1 - \exp(\ln(1 - z))$ .

36. Show that the series

$$\sum_{n=1}^{\infty} \frac{z}{(z - n)^2}$$

converges to a complex differentiable function in the domain

$$C - \{1, 2, \dots, n, \dots\}.$$

37. Show that  $\exp(z) = \lim_{m \rightarrow \infty} (1 + z/m)^m$ . (Hint: Compute the power series expansion of  $(1 + z/m)^m$ .)

38. Show that a real polynomial of odd degree always has a real root.

39. Let  $\omega = \exp(2\pi i/n)$ . Show that

$$1 - z^n = (1 - \omega z)(1 - \omega^2 z) \cdots (1 - \omega^{n-1} z)$$

40. Let  $P$  be a polynomial. Show that  $P$  is the square of another polynomial if and only if every root of  $P$  occurs with even multiplicity.

41. If  $P, Q$  are two polynomials,  $P$  divides  $Q$  if and only if every root of  $P$  is a root of  $Q$  with no larger multiplicity.

42. Suppose  $P$  is a polynomial of degree at least two. Show that there is a  $c$  such that  $P(z) - c = 0$  has at least one multiple root. (*Hint*: Consider  $P'$  as defined in Problem 10. If  $P'(a) = 0$ , take  $c = P(a)$ .)

43. Let  $f_1, \dots, f_n$  be functions in  $C(X)$ . Show that these functions are independent if and only if there are points  $x_1, \dots, x_n$  such that the matrix  $(f_i(x_j))$  is nonsingular.

44. Show that the functions  $e^{rx}, xe^{rx}, \dots, x^n e^{rx}$  are independent.

45. Show that if  $P, Q$  are polynomials,  $S(L_P) \subset S(L_Q)$  if and only if  $P$  divides  $Q$ .

46. If  $P$  is a polynomial of degree at least two, there is a  $c \in C$  such that the equation  $L_P f = cf$  has a solution of the form  $xe^{rx}$ .

47. If a linear differential equation has polynomial coefficients, it has global solutions on all of  $R$ .

48. Let  $\{c_n\}$  be a sequence of complex numbers such that  $\sum |c_n| < \infty$ . Let  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ . Prove that  $|c_0|$  is *not* a relative maximum of  $|f|$ , unless all other coefficients vanish.

49. Suppose the function

$$f(x, y) = x^2 - y^2 + iv(x, y)$$

is complex differentiable. Find  $v$ .

50. If  $f$  is a polynomial in  $x, y$  which is complex differentiable, then  $f(x, y)$  has the form  $Q(x + iy)$ , where  $Q$  is a polynomial. (*Hint*: Substitute  $x = (z + \bar{z})/2$ ,  $y = (z - \bar{z})/2$ , and use the Cauchy-Riemann equation.)

51. Suppose  $f$  is a  $C^2$  complex-valued function defined on a domain  $D$  in  $C$ . Show that if  $f$  and  $fz$  are both harmonic, then  $f$  is complex differentiable.

52. If  $f, g$  are complex differentiable and  $|f|^2 + |g|^2$  is constant, then both  $f, g$  are constant.

53. Suppose that  $f$  is a one-to-one mapping of a domain  $D \subset C$  onto  $\Delta \subset C$ . Let  $g: \Delta \rightarrow D$  be the inverse of  $f$ . Show that if  $f$  is complex differentiable, so is  $g$ .

54. We may consider the function  $e^z$  as a mapping from the plane to the plane. Let  $z = x + iy$ ,  $u = \operatorname{Re} e^z$ ,  $v = \operatorname{Im} e^z$ ; that is

$$u = e^x \cos y \quad v = e^x \sin y$$

(a) Show that this mapping maps the lines  $x = \text{const.}$  on the circles centered at the origin, the lines  $y = \text{const.}$  go onto the rays through the origin.

(b) Show that in any interval  $\{a < y < a + 2\pi\}$  this mapping takes every value precisely once.

(c) In particular,  $e^z$  maps the horizontal strip  $\{-\pi < y < \pi\}$  one-to-one onto the entire plane except for the negative real axis. Call this domain  $D$ . Define the complex logarithm  $\log z: D \rightarrow \{-\pi < y < \pi\}$  as the inverse of this mapping. Show that  $\log z$  is complex differentiable and

$$(\log)' z = \frac{1}{z}$$

Show also that  $\log z$  can be represented by a power series centered at 1 in the disk  $\{|z - 1| < 1\}$ . (Recall Miscellaneous Problem 35.) Notice that this provides a way for extending real functions to the complex domain besides that of power series. For example, the power series expansion about 1 of  $\log x$  extends it only to the unit disk centered at 1. The above extension of  $\log$  is defined in the entire plane except the negative real axis. This process is called *analytic continuation*.

55. Consider  $z^2$  as a mapping of the plane into the plane. Show that it maps the open right half plane one-to-one onto the domain  $D$  of Problem 54. Let  $\sqrt{z}$  be the inverse, and show that  $\sqrt{z}$  is complex differentiable. Provide a similar discussion for the mapping  $z^n$ .

56. Discuss the mapping properties of  $\cos z$ ,  $\sin z$ .

57. Show that the power series expansion of the solution of Exercise 8(d) with initial values  $y(0) = 1$ ,  $y'(0) = 0$  does *not* converge outside the unit disk.

58. Suppose that  $f, g$  are complex-valued functions defined on the interval  $I$ . Show that

$$\int_I \frac{f(t)}{z - g(t)} dt$$

is complex differentiable and can be represented by a power series at any point of the image of  $g$ .

59. If  $f$  is  $C^1$  in  $C \times X$  and for each fixed  $x$ ,  $f(z, x)$  is differentiable in  $z$ , then

$$F(t) = \int f(z, x) dx$$

is also complex differentiable.

60. The equation of Exercise 6 is called *Legendre's equation* and the solutions  $\{f_k\}$  for integral  $k$  are called the *Legendre polynomials*. They have this interesting property:

$$\int_{-1}^1 f_m(x) f_n(x) dx = 0 \quad \text{if } m \neq n$$

To prove this we must observe that Legendre's equation may be written as

$$((1-x^2)y')' + k(k+1)y = 0$$

Thus

$$\begin{aligned} \int_{-1}^1 f_m f_n &= \frac{-1}{m(m+1)} \int_{-1}^1 [(1-x^2)f_m'(x)]f_n'(x) dx \\ &= \frac{1}{m(m+1)} \int_{-1}^1 (1-x^2)f_m'(x)f_n'(x) dx \end{aligned}$$

by integration by parts. Now do the same, interchanging  $m$  and  $n$ .

61. Let  $P$  be a polynomial of degree  $d$ . Show that  $f(z) = e^{P(z)}$  is complex differentiable. Show that  $f^{(n)}(z)e^{-P(z)}$  is a polynomial of degree  $n(d-1)$ .

62. Show that the polynomial

$$\exp(x^2) \frac{d^n}{dx^n} (\exp(-x^2))$$

solves the differential equation  $y'' - 2y' + 2ny = 0$ .

63. (a) Find a  $C^\infty$  real-valued function  $f$  defined on  $R^n$  with these properties:

- (i)  $0 \leq f(x) \leq 1$
- (ii)  $f(x) > 0$  if  $\|x - x_0\| < R$
- (iii)  $f(x) = 0$  if  $\|x - x_0\| \geq R$

(By  $C^\infty$  we mean all higher-order partial derivatives exist and are continuous.)

(b) Let  $X$  be a closed set in  $R^n$ , and suppose  $B_1, \dots, B_n$  are balls in  $R^n$  such that  $X \subset B_1 \cup \dots \cup B_n$ . A *partition of unity on  $X$  subordinate to  $B_1, \dots, B_n$*  is a collection  $\{f_1, \dots, f_n\}$  of  $C^\infty$  functions such that

- (i)  $0 \leq f_i \leq 1$
- (ii)  $f_i(x) = 0$  if  $x \notin B_i$
- (iii)  $\sum_{i=1}^n f_i(x) = 1$  if  $x \in X$

Find such a partition of unity.